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A SECOND BOOK IN ALGEBRA

BRIEFER EDITION

BY

FLETCHER DURELL, PH.D.
HEAD OF MATHEMATICAL DEPARTMENT,
THE LAWRENCEVILLE SCHOOL

AND

E. E. ARNOLD, M.A.
SUPERVISING PRINCIPAL, THE PUBLIC SCHOOLS OF
THE PELHAMS, NEW YORK,
FORMERLY SPECIALIST IN MATHEMATICS, THE
UNIVERSITY OF THE STATE OF NEW YORK

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PREFACE

THE present volume contains a second course in algebra adapted to the latter part of the High School curriculum. Many schools give only one half year to the study of the second course in algebra, and it is the object of this book to supply material adapted to the needs of such schools. In different parts of the country, tentative syllabi have recently been worked out for such a briefer study, and these syllabi have been carefully considered by the authors in writing the present volume.

The features which characterize the authors' FIRST BOOK IN ALGEBRA are continued and developed. Extended use is made of the *graph,* among other advances being the per capita and efficiency curves and dissected bar graphs.

In the treatment of the *formula,* the pupil is asked more frequently to supply letters and to devise the complete formula.

No other part of algebra so develops thought power and cultivates an appreciation of the spirit of algebra as does the *verbal problem.* Accordingly, more difficult and varied verbal problems are supplied, many being given in the inverse form, where the conditions of a problem are stated in symbols and the pupil is asked to convert the symbolic statement into general language.

Throughout the book, *pivotal and permanently valuable number facts, and laws* from other branches of study are introduced in various ways. This gives a correlation of algebra with geography, history, economics, and other school studies. Also the numerous examples where a pupil is asked to reduce units of different orders to a single unit constitute a useful preliminary to the study of physics and engineering.

The *self-activity* of the pupil is aroused and developed by examples which require him to invent and solve problems to meet given sets of conditions.

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A SECOND BOOK IN ALGEBRA

BRIEFER EDITION

CHAPTER I

FUNDAMENTAL PROCESSES

1. Some Algebra Which You Have Already Learned. In a former course in algebra you have learned that by an extended use of symbols in dealing with numbers, much labor is often saved, and sometimes also results are obtained which could not have been obtained in any other way. It is the object of this second course in Algebra to develop these advantages still further.

First of all, it will be useful to recall to mind some of the things learned in the first course in algebra.

The following illustration will bring to mind one of the first and most important ways in which labor is saved by the use of symbols in algebra. The rule for obtaining the interest on a given sum of money for a given time at a given rate is as follows: Multiply together the principal, the rate per cent expressed decimally, and the number of years. By the use of symbols, this rule is expressed as a formula, thus: $i=prt$. By this means, the 72 letters in the rule are reduced in the formula to 5 symbols.

Also from $i=prt$, we obtain

$$t=\frac{i}{pr},\quad r=\frac{i}{pt},\quad \text{and}\quad p=\frac{i}{rt}.$$

Express each of these results as a rule. Count the number of letters in each of these rules. How many letters are in all four of

these rules taken together? Hence, $i = prt$ takes the place of how many letters in all?

EXERCISE 1

By use of the formula $i = prt$, solve Exs. 1–4.

1. Find the interest on $350 for 2 yr. 6 mo. at 5 per cent.

2. In how many years will the interest on $250 at 6 per cent be $52.50?

3. Find the rate per cent at which the interest on $500 in 3 yr. 6 mo. is $70.

4. On a given principal, the interest in 1 yr. 9 mo. at 6% is $78.75. Find the principal.

5. In the formula $p = br$, let $p = 32$ and $b = 800$, and find r. State this example as a problem in percentage.

6. In the formula $a = lw$, let $a = 903$ sq. ft. and $w = 21.5$ ft., and then find l. State this example as a problem concerning the area of a rectangle.

7. If 40 cu. ft. of a certain kind of coal make a ton, find the formula for the numbers of tons (T) that may be put into a bin l ft. long, w ft. wide, and d ft. deep.

8. By use of the formula of Ex. 7, find how deep a bin must be to contain 5 tons of coal, if this bin is to be 10 ft. long and 5 ft. wide.

9. The formula $p = br$ takes the place of what three rules? How many letters are in these three rules? How many symbols are in the formula $p = br$? Compare these two numbers.

10. Find a formula for the length in feet (L) of a shelf to hold a books each t_1 inches thick, b books each t_2 inches thick, and c books each t_3 inches thick.

11. By use of the formula of Ex. 10, find the length in feet of a shelf long enough to hold 5 books each 2 in. thick, 6 books each $1\frac{1}{2}$ in. thick, and 8 books each $1\frac{3}{4}$ in. thick.

EXERCISE 2

By the aid of symbols, write the following in the shortest way you can:

1. 5 plus x minus a.

2. The product of 7, x, and y.

3. x divided by the sum of a and b.

4. p divided by the product of q and r.

5. The product of 5 and a, minus 3 times the product of x and y.

6. The square of a minus b equals the square of a, minus twice the product of a and b, plus the square of b.

7. Write three consecutive numbers, the smallest of which is n. Also three, the largest of which is n. Also three, the middle one of which is n.

8. Charles has x dollars, and Henry has 5 less than twice as many. How many has Henry? How many have both together?

9. What is the cost in cents of 5 lb. of sugar at p cents a pound and 3 lb. of coffee at q cents a pound? Also express the cost in dollars.

EXERCISE 3

When $a=5$, $b=3$, $c=1$, $x=6$, find the numerical value of

1. $3a+2(x-c)$.

2. $(3a+2)(x-c)$.

3. $3c^2+7a-5(x-b)$.

4. $\dfrac{3x^2-b^2}{a-2c}$.

5. $\dfrac{(x-c)(2b-a)}{3(b^2+c)-2a}$.

6. $7x(5a-4x)-cx^2$.

7. $3x(x-a)(x^2-ab^2+2ac^2)$.

If $a=4$, $b=\frac{3}{2}$, $c=0$, $x=1$, $y=9$, find the value of

8. $\sqrt[3]{2ax}$.

9. $\sqrt{ax^2y}$.

10. $3b\sqrt{6b^3x^3}$.

11. $3cx\sqrt{a^2+bc^2}$.

12. $3x\sqrt{aby-5x}-b^2c^2$.

If $a = 1.23$, $b = .56$, $c = .04$, and $d = .3$, evaluate (that is, find the numerical value of) to the nearest thousandth:

13. $2a(b-c)(a^2+d^2)$.

14. $(a+b) \div (c+d)$.

15. $\frac{abc}{5d} + \left(\frac{a}{b} + \frac{c}{d}\right) - \frac{4d}{a+2b}$.

16. Find the difference between $6abx - 2ab$, and $6abx \div 2ab$, when a, b, and x equal 3, 5, and 6, respectively.

EXERCISE 4

1. What is a formula? Give two examples of a rule and the corresponding formula.

2. Give a definition of Algebra.

3. What is an algebraic expression? Write an algebraic expression containing three terms.

4. What is a term? Name the three terms in the expression which you wrote in answering Ex. 3. Why is it customary to regard an algebraic expression as made up of terms?

5. What is a coefficient? Give an example of a numerical coefficient. A literal coefficient. A mixed coefficient.

6. What is an exponent? Write a term containing both the exponent 2 and exponent 3.

7. If a letter is written without an exponent, what exponent is the letter supposed to have?

8. Explain the difference between a power and an exponent.

9. By making selections from the following list of terms, $5x^2$, $-3xy$, $7(a-b)$, $4a^3$, $8abc^2$, $-5(x+y)$, write (1) three monomials; (2) three trinomials; (3) four binomials; (4) four polynomials, each of which contains a different number of terms from the rest.

10. What are similar terms? Write two similar terms each of which contains the exponent 3.

11. Write two terms which are not similar, but each of which contains the letters x and y and no other letters, and the exponents 2 and 3 but no other exponents.

12. What are the factors of an expression? What are the factors of 15? Of $5\,xy$? Of $5\,x^2y^3$? What is meant by literal factors? Give an example of a term containing three literal factors.

13. What is meant by the degree of a term? Write a term of the 2d degree containing only one letter. Also a term of the 2d degree containing two letters.

14. Write a term of the 3d degree containing two and only two letters. Also one of the 8th degree containing three and only three letters.

15. Write a polynomial of the 4th degree containing five terms arranged according to the ascending powers of some one letter. Rewrite it in the descending order.

EXERCISE 5

1. Define negative number. Give three examples of a positive number with the corresponding negative number.

2. Give three events which have negative dates with respect to the birth of Christ.

3. Give three events which have negative dates with respect to July 4, 1776.

4. At a certain place during a given year, the highest temperature was 99° and the lowest was $-14°$. Find the difference between these two temperatures.

5. The highest point of land on the earth's surface is the top of Mt. Everest, 29,002 ft. above sea level; the lowest point of dry land is in the valley of the Jordan, 1290 ft. below sea level. What is the difference in altitude between these places?

6. Count the number of letters and symbols in the phrase "7 degrees below zero." Also in "$-7°$." Compare the two numbers.

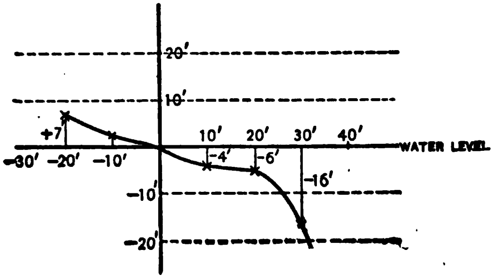
7. On the foot rule, show the sum of 2 in., -3 in., and 7 in. If each inch represents 10 yd., what would this example mean in terms of advances in a football game?

8. Find the algebraic sum of $7\frac{3}{4}-2\frac{1}{2}-1\frac{1}{8}$. What is meant by an algebraic sum?

9. How much money must be added to $-\$213$ to make $920?

10. Define algebraic numbers. What is meant by the absolute value of a number? What is the absolute value of -20 in.?

11. At a place where a stream was 110 ft. wide, the depth of the water was measured at intervals of 10 ft. These depths were found to be in feet as follows: 4, 6, 16, 23, 22, 14, 11, 7, 4, 2. Also 10 ft. from the left shore, the height of the bank above water level was 3 ft., and at 20 ft. from the left shore it was 7 ft. Similarly at 10 and 20 ft. from the right shore, the heights of the land were 5 ft. and 8 ft. respectively. Make a drawing of the banks and the bottom of the stream as determined by these numbers.



Sug. The beginning of the drawing is shown in the above diagram. Let the pupil complete the drawing.

EXERCISE 6

Add and check each result:

1.	2.	3.	4.	5.
-5	$-3\,ab^2$	$-2.5\,a^2x^3$	$-4(a-b)$	$\frac{3}{4}\,\pi r^2$
27	$-5\,ab^2$	$-1.7\,a^2x^3$	$-7(a-b)$	$-\frac{1}{6}\,\pi r^2$
-15	$12\,ab^2$	$.9\,a^2x^3$	$(a-b)$	$-\frac{3}{2}\,\pi r^2$
-3	$-ab^2$	$-.3\,a^2x^3$	$6(a-b)$	$\frac{1}{4}\,\pi r^2$
7	$-3\,ab^2$	$.5\,a^2x^3$		

6. $2\,a+b,\ 3\,a-c,\ 2\,b+3\,c.$ 7. $ax^2y,\ bx^2y,\ -cx^2y.$

8. $\frac{1}{2}\,x-\frac{3}{4}\,y+z-\frac{1}{2}\,y+3\frac{1}{2}\,z+\frac{5}{6}\,x+\frac{2}{3}\,x-\frac{1}{2}\,z+\frac{7}{6}\,y-3\frac{1}{2}\,x-7\frac{1}{4}\,z+2\frac{1}{2}\,y.$

9. $3\,ab+.2\,ac+.75\,bc-.02\,ab-.35\,ac+.003\,bc+.6\,ab-7.2\,ac.$

10. $a^{3n}+5\,a^{2n}-6\,a^n,\ 4\,a^{3n}-2\,a^{2n}+3\,a^n,\ -2\,a^{3n}-a^{2n}+3\,a^n.$

11. $2\,n-1$ and $3-n-2\,n^2.$

12. Reduce $5\,aabbb-3\,aabbb-7\,aabbb+8\,aabbb$ to its simplest form. Compare the number of symbols in the two forms.

13. Add in the shortest way $6\frac{7}{8}-3\frac{1}{2}+\frac{1}{8}+2\frac{1}{2}+4\frac{1}{2}-3.$ Make up and work an example involving 16th's and 4th's.

14. Add $5\,h+3\,t+u,\ 2\,t+3\,u,\ 2\,h+u.$ If h means hundreds, t tens, and u units, express the example as an addition of arithmetical numbers.

15. Where $x=3$, and $y=5$, find the value of $8\,x^2y-9\,x^2y+6\,x^2y-2\,x^2y$, (1) without first adding the terms; (2) after adding the terms. Compare the amount of work in the two processes. What does this comparison teach us as to the efficiency value of addition in algebra?

16. Make up and work an example in which three binomials and two trinomials are added together.

EXERCISE 7

Subtract and check each result:

$$\begin{array}{cccc} \textbf{1.} & \textbf{2.} & \textbf{3.} & \textbf{4.} \\ a^2-3ab+2b^2 & 9(x-y) & -7\sqrt{a-x} & a+b \\ \underline{3a^2-5ab-7b^2} & \underline{-2(x-y)} & \underline{\sqrt{a-x}} & \underline{\quad b+c} \end{array}$$

$$\begin{array}{ccc} \textbf{5.} & \textbf{6.} & \textbf{7.} \\ (2a-3b)x & (a+b-c)x & (3a-2b-c)y \\ \underline{(a+b)x} & \underline{(b+c-a)x} & \underline{(5a-2b+d)y} \end{array}$$

8. Subtract $2a+b$ from $3a+c$.

9. From $n-2$ subtract 3. **10.** Subtract bx^3 from ax^3.

11. Subtract $(a^2-2ab+b^2)x$ from $(a^2-b^2)x$.

Find the expression which must be added

12. To $2a-3b$ to make $5a-7b$.

13. To $3a^2-2a+1$ to make $7a^3-2a^2-3$.

14. To $x^{2n}+5x^n+7$ to make $8x^{2n}-3x^n-2$.

15. From the sum of $3a+5b-2c$ and $7a-2b+5c$ subtract their difference.

16. From the sum of $(5a-3b)x$ and $(2a-5b)x$ subtract their difference.

If $A=3x^2-2x+5$, $B=5x^2-7$, $C=3x-2$, find

17. $A+B-C$. **18.** $B-A-C$. **19.** $C-A-B$.

20. If Rome was founded in the year -753 and captured by the Goths in the year $+410$, find the number of years between these dates.

21. Hydrogen melts at a temperature of $-256°$ C. and ammonia at $-34°$ C. Which melts at the higher temperature? Find the difference between the two temperatures.

22. Subtract $2h+3t+5u$ from $7h+5t+6u$. If h means hundreds, t tens, and u units, express this example as a subtraction of arithmetical numbers.

23. What is the parenthesis sign? Write three symbols which may be used instead of the parenthesis sign.

24. What is the meaning of the parenthesis sign? Give an example showing its use.

Simplify:

25. $15-(-6.4)+(-5.8)$. **26.** $2.7+\{.4-(3.07-.015)\}$.

27. $5x+\{-2x-(.3x-2.5)-4.7\}$.

28. $6a^2-[(3a-5)-7a^2]-(2a^2-7)$.

29. In $7x-(3a-2b+c)$ what is the sign of $3a$ as the example stands?

In the next four examples, inclose the last three terms in a parenthesis preceded by a minus sign.

30. $3x^2-a^2+2ab-b^2$. **31.** $5x^2-1-2a-a^2$.

32. $5a^2+2x^2-3xy-y^2$. **33.** $7a-3b-4c-x+y$.

34. State which of the following are identical with $x-y$; viz., $y-x$, $-(x+y)$, $-(-x+y)$, $-y+x$.

35. Make up a similar example concerning $x+y-z$.

36. Make up and work an example in which a trinomial is subtracted from a binomial.

EXERCISE 8

1. Solve the equation $8x-72=3x+23$, (1) without transposition of terms; (2) using transposition.

2. What does the word *transposition* mean as used in algebra? What are the advantages of employing transposition in solving equations? When are the advantages in the use of transposition most apparent?

Solve and check:

3. $5x-.108=1.4x$. **4.** $5x-(2x-3)=6$.

5. $10-(3x-5)-[8-(7x+2)]=0$.

6. $14x-[22.4+(58.8-16x)]=2.5$.

7. $11r = 42.9 - [-(3r - 38.4) + 6.5]$.

8. Give a definition of an equation.

9. What is a root of an equation? Is 2 a root of the equation, $5x + 4 = 7x - 3$? What is its root?

10. What is meant by solving an equation?

11. Does $-.2$ satisify the equation $3x - 1.5 - (x - 4) = 8.2$?

EXERCISE 9

ORAL

1. Express each of the following statements in symbols: x exceeds y by 3; (2) x is greater than y by 3; (3) y is 3 less than x; (4) x equals y increased by 3; (5) y equals x diminished by 3. How many of the preceding statements express the same fact?

2. In four different ways express in words $a - b = 4$.

3. The dimensions of a given rectangle are x and y. State the dimensions of a rectangle which exceed those of the given rectangle by 40 %.

4. A man bought a house for x dollars and sold it so as to gain a dollars. For how many dollars did he sell it? What per cent did he gain?

5. At a cents a yard, how many yards of calico can be obtained in exchange for b dozen eggs worth c cents a dozen?

6. State the interest on $2p$ dollars at r per cent for $3t$ years.

7. What is the cost of a articles bought at c cents per hundred?

8. If x represents a man's present age in years, what does $x - 10$ represent? $x - 5$? $2x$?

9. If c represents the cost of 1 pound of sugar, what does $\frac{11}{2}c$ represent?

10. If x and y are the dimensions of a rectangle, what does $(x - 2)(y - 3)$ represent?

11. If a man bought a house for b dollars and sold it for s dollars, what does $s - b$ represent?

12. A man had a journey of 150 mi. to go. He has now traveled for h hours at m miles an hour. What is the meaning of $150-hm$?

13. A boy has m marbles. He loses $\frac{2}{3}$ of these and then buys $\frac{3}{4}$ as many as he had at first. How many does he then have?

14. A train runs m miles per hour. How far will it go in c minutes?

15. Find the difference in area between a surface a feet square and one containing a square feet.

EXERCISE 10

1. Four partners together made $28,500 in one year. The first partner received $2500 more than the third, the second $1000 more than the third, and the fourth twice as much as the third. How much did each receive?

SUG.

NUMBERS DEALT WITH	SYMBOLS
Number of dollars received by first partner	$=x+2500$.
Number of dollars received by second partner	$=x+1000$.
Number of dollars received by third partner	$=x$.
Number of dollars received by fourth partner	$=2x$.
Number of dollars received by all together	$=28,500$.

EQUALITY AMONG NUMBERS

Sum of first four numbers = last number.

Let the pupil complete the solution.

2. Two girls made $61.80 by keeping a refreshment stand. The girl who supplied the materials received $20 more than the other. How much did each receive?

3. A man and a boy together made $106 from a garden. The man received $25 more than twice what the boy received. How much did each make?

4. Three partners together made $13,000 in one year. The first received twice as much as the third, and the second received $1000 more than three times as much as the third. How much did each receive?

5. Find four consecutive numbers whose sum is 74.

6. In a certain kind of concrete the parts of cement, sand, and gravel are proportional to 1, 3, 5. How many cubic feet of each are needed in making 1800 cubic feet of concrete?

7. One number exceeds three times another number by .23. The sum of the numbers is 1.04. Find the numbers.

8. In a certain election 957 votes were cast. The successful candidate came within 18 votes of receiving twice as many as the defeated candidate. How many votes did each receive?

9. If three times a certain number is increased by 17 the result is 6 less than four times the number. Find the number.

10. For every dime of his savings that a boy spent for books his father gave him a quarter to spend for the same purpose. If the boy spent $78.75 in all, how much did his father give him?

11. By what number must $c+d$ be multiplied in order that when b is taken from the result, the remainder will be $c-d$.

12. The perimeter of a rectangle is 176 in. The length is 8 in. less than three times the width. Find the area of the rectangle in square feet.

2. Example in Multiplication.

Ex. Arrange and multiply $x^3-2a^3+3a^2x$ by $2a^2+x^2-ax$.

$$\begin{array}{rllllll}
 & x^3 & +3a^2x & -2a^3 & & & =2\\
 & x^2 & -ax & +2a^2 & & & =2\\ \hline
 & x^5 & & +3a^2x^3 & -2a^3x^2 & & \\
 & & -ax^4 & & -3a^3x^2 & +2a^4x & \\
 & & & +2a^2x^3 & & +6a^4x & -4a^5\\ \hline
\textit{Product,} & x^5 & -ax^4 & +5a^2x^3 & -5a^3x^2 & +8a^4x & -4a^5=4
\end{array}$$

3. Law of Exponents in Multiplication.

By definition of a power, $a^5 \times a^2 = (a \cdot a \cdot a \cdot a \cdot a)(a \cdot a) = a^7$.

Similarly, when m and n are any positive whole numbers,
$a^m \times a^n = (a \cdot a \cdot a \ldots$ to m factors$)(a \cdot a \cdot a \ldots\ldots$ to n factors$)$,
$= a \cdot a \cdot a \ldots$ to $(m+n)$ factors $= a^{m+n}$.

Hence, $$a^m \times a^n = a^{m+n}.$$

Ex. Multiply $3x^{2n-3}$ by $4\,x^{n+5}$.

Since $2n-3$ and $n+5$ added, give $3n+2$,
the product is $12\,x^{3n+2}$. *Ans.*

EXERCISE 11

Multiply:

1.	2.	3.	4.	5.	6.
$4\,x^2$	2^{n-1}	ax^{n-1}	$3\,a^2x^{n-1}$	$-5\,ay^{n-3}$	x^{2n}
$.2\,x^3$	2^3	x^3	$-4\,ax^4$	$.2a^2y^{n-1}$	x^{3n}

Multiply and check the following, arranging the terms of each polynomial in ascending or descending order:

7. $3\,x+1$ by $x+2$. 8. $3\,x+4\,y$ by $4\,x-5\,y$.

9. $x^2-3\,x+1$ by $2\,x-3$.

10. $3\,a^2-4\,ax+x^2$ by $2\,a-3\,x$.

11. $2\,y^3-4\,y^2+y-1$ by $2\,y-5$.

12. $4\,x^2+x-2$ by $3\,x^2-x-5$.

13. $2\,x^3+3\,x^2y-4\,xy^2+y^3$ by $3\,x^2+y^2$.

14. $5\,x^3+x-5$ by $3\,x^2-2\,x-4$.

15. $4\,a^3+2a^2b-b^3$ by $3\,a^3-ab^2+b^3$.

16. $2\,x^4-4\,x^3+3\,x^2-2\,x+1$ by $1+2\,x+x^2$.

17. $3\,x^2+6x+x^3+15$ by $1+x^2-3\,x$.

18. $\frac{1}{4}\,a^2+\frac{1}{6}\,ab+\frac{1}{9}\,b^2$ by $\frac{1}{2}\,a-\frac{1}{3}\,b$.

19. $1.2\,x^2+1.5x+6.4$ by $2.4\,x-3$.

20. $1.8\,a^2-3.2\,a+.48$ by $2.5\,a+.5$.

21. $a^n+2\,a^{n-1}+3\,a^{n-2}$ by $3\,a^2-2\,a+5$.

22. $x^{n+1}-3x^{n}+4x^{n-1}-5x^{n-2}$ by $x^{n}+2x^{n-1}$.

23. $4x^{n+1}-3x^{n}+x^{n-1}$ by $x^{2}+2x+1$.

24. What different ways are there of checking a result in multiplication?

Multiply and check each result:

25. $5(x+y)^{2}-4(x+y)+2$ by $3(x+y)$.

26. $5(x-y)^{2}+3(x-y)-2$ by $2(x-y)-5$.

Simplify and check:

27. $a-2(a-3)$. **30.** $(3x-2)(5x^{2}-3x-2)$.

28. $(a-2)(a-3)$. **31.** $(3x+5)(x+y)-y$.

29. $3x-2(5x^{2}-3x-2)$ **32.** $x-2(x-3)(x-5)$.

33. $5-3(a-2)^{2}-2(3-2a)(1+a)$.

34. $3a^{2}-[x(a-x)-a(x-a)]-2x^{2}$.

35. $3[x(.2-x)-.5(x-4)]-1.6x^{2}$.

36. In the shortest way, multiply the sum of $(x-2y)^{2}$ and $(2x-y)^{2}$ by $3x-2(x-y)$.

If $x=3$, $y=0$, $a=-2$, $b=-5$, find the value of

37. $b^{2}y+3a(a-b)$. **38.** $4a^{2}-axy(4a-b)$.

39. $2(a^{2}+b)-bxy+a^{3}x$.

40. $3x(x-2a)-\{x-(x-1)(a+1)-(a+x)^{2}\}+5ax$.

41. From 3 times the product of $x+5$ and $3x-.5$ subtract 5 times the product of $1-2x$ and $2x-3$.

Solve and check:

42. $3x^{2}-2(x+1)(x-5)=(x+3)(x-1)$.

43. $(5-3x)(3+4x)-(7+3x)(1-4x)+1=0$.

44. $2[x+x(x-3)+1]-(2x+5)(x-1)=0$.

45. $4(28.5-3x)-3(2x-11.3)+4.5=2(48.7-5x)-3.4$.

4. Example in Division.

Ex. Arrange and divide $11x-3x^{2}-6+6x^{4}+7x^{3}$ by $3x-2+2x^{2}$.

$$\begin{array}{r|l}
\overbrace{6x^4+7x^3-3x^2+11x-6}^{\textit{Dividend}} & \overbrace{2x^2+3x-2}^{\textit{Divisor}}=15\div 3 \\
6x^4+9x^3-6x^2 & \underbrace{3x^2-\ x+3}_{\textit{Quotient}}=\ 5 \\
\hline
-2x^3+3x^2+11x-6 & \\
-2x^3-3x^2+\ 2x & \\
\hline
6x^2+\ 9x-6 & \\
6x^2+\ 9x-6 & \\
\hline
\end{array}$$

5. Law of Exponents in Division.

If m is greater than n, by § 3

$$a^{m-n}\times a^n=a^{m-n+n},$$

or,
$$a^{m-n}\times a^n=a^m.$$

Dividing each of these equals by a^n,

$$a^m\div a^n=a^{m-n}.$$

Ex. Divide $12\,a^{2m-3}$ by $3\,a^{m-1}$.

$$\frac{12a^{2m-3}}{3\,a^{m-1}}=4\,a^{m-2}.\quad \textit{Quotient.}$$

EXERCISE 12

Divide and check:

1. $-30\,x^3y^2$ by $-6\,x^2y$.
2. $.4\,ax^2$ by $.8\,x^2$.
3. $8\,a^{3n}$ by $2\,a^n$.
4. $15\,x^{2n+5}$ by $-3\,x^{n-3}$.
5. $2\,a^2x^{m+n}$ by ax^n.
6. a^{m+p} by a^{m+q}.
7. $-1.4(a-b)^5$ by $-7(a-b)^3$. By $-2(a-b)^4$.
8. a^{6n} by a^{2n}. By a^{3n}. By $-a^n$.
9. $-6\,a^{n+3}$ by $2\,a^{n+1}$. By a^{n+2}. By $-3\,a^{n-1}$. a^{2n-3}.
10. $6\,x^3-19\,x^2y+21\,xy^2-10\,y^3$ by $3\,x-5\,y$.
11. $2\,x^5-5\,x^4-2\,x^3+9\,x^2-7\,x+3$ by $2\,x^2+x-3$.
12. $x^3+8\,y^3$ by $x+2\,y$.
13. $16\,a^4-1$ by $2\,a-1$.
14. $4\,a^2+b^2+x^2-4\,ab+4\,ax-2\,bx$ by $2\,a-b+x$.

15. $\frac{1}{2}x^3-17\frac{2}{3}x+2$ by $\frac{3}{4}x+\frac{9}{2}$.

16. $x^4-\frac{5}{6}x^3+\frac{19}{36}x^2-\frac{5}{36}x+\frac{1}{36}$ by $x^2-\frac{1}{3}x+\frac{1}{9}$.

17. $.1\,a^2-.23\,ab+.12\,b^2$ by $.2\,a-.3\,b$.

18. $4.5\,x^3-7.1\,x^2-.4\,x+.24$ by $1.8\,x^2-3.2\,x+.48$.

19. $x^{n+1}-x^{n-1}-6\,x^{n-2}-2\,x+4$ by $x-2$.

20. $6\,x^{2n+1}-13\,x^{2n}+6\,x^{2n-1}$ by $3\,x^{n+1}-2\,x^n$.

21. Make up and work an example in which a binomial of the third degree is divided by a binomial of the first degree without a remainder.

Divide:

22. $12(x+y)^4-8(x+y)^3-16(x+y)^2$ by $4(x+y)^2$.

23. $9(a-b)^5-12(a-b)^3+15(a-b)$ by $3(a-b)$.

If $x=0$, $y=2$, $z=-3$, $a=1$, find the value of

24. $3\,a^2y$.

25. $3\,a^3x$.

26. $5+a^2x$.

27. $4\,x^2z+5\,az^2$.

28. $\dfrac{5\,x-3\,a^2z}{4}$.

29. $\dfrac{3\,a^5y+z^3}{x+y}$.

30. $y+5\,x$.

31. $\dfrac{x}{a}+z^2$.

32. $xz^2+5\,a^4$.

EXERCISE 13

ORAL

1. How many cubic feet are in a box l feet long, w inches wide, and d inches deep?

2. How many feet are in the circumference of a circle whose radius is a inches? How many square feet in the area of the circle?

3. How many square feet are in the surface of a cube whose edge is i inches?

4. If the area of a rectangle is r square feet and the rectangle in i inches wide, express the length of the rectangle in feet.

5. The volume of a box is to be c cubic feet. If the top of the box is a square whose side is a inches, express the depth of the box in feet.

6. If the edges of a box are l, w, and h feet, express the sum of all the edges. Express also the area of the surface, omitting the top.

7. Four boys share equally a pounds and b ounces of candy. How many ounces does each receive?

8. A building lot is x feet square. How much will it cost to fence it at b cents a yard?

9. Find the value in dollars of a block of marble, m feet square and x feet high, at c cents a cubic foot.

10. How many square yards are in the surface of a wall l feet long and h feet high, if d square feet are deducted for door and window spaces?

11. If a cistern contains g gallons of water, and a quarts per minute run into it, how many gallons will it contain at the end of an hour?

12. A boy had d dollars and spent c cents a day. How many dollars did he have left at the end of t days?

EXERCISE 14

1. Allowing 34 sq. ft. for a roll, find a formula for the number of rolls (r) of paper required to cover the walls and ceiling of a room whose dimensions are l, w, and h feet, if d square feet are deducted for doors and windows.

2. If 7½ gallons be taken as equal to one cubic foot, and the dimensions of a rectangular cistern are a, b, c feet, find a formula for G, the number of gallons the cistern will hold. By use of this formula solve Ex. 3.

3. If a cistern is to be 8 ft. long and 6 ft. wide, how deep must it be to hold 204 gallons?

4. If a house whose value is h dollars is insured for the a/bth part of its value, and the insurance rate is r, obtain a formula for P, the insurance premium.

5. If a man's salary the first year is a dollars and he receives an increase of b dollars a year, obtain a formula for his salary (s) at the end of t years.

6. If a man's salary the first year is \$1500 and he receives an annual increase of \$125, how many years will

it be before his salary is $3000? Solve by use of the formula obtained in Ex. 5.

7. If two cars start at the same place and move in opposite directions at the rate of a and b miles an hour respectively, obtain a formula for D, the number of miles they will be apart in t hours. Make up and solve a numerical example by use of this formula.

8. The rule for determining the length of time a beef roast should be cooked is as follows: to 20 minutes add one-quarter of an hour for each pound in the roast. From this rule, obtain a formula for the number of hours (H) a roast weighing n pounds should be cooked.

By use of this formula find the number of hours a roast weighing $11\frac{1}{2}$ lb. should be cooked.

9. A man had n children, and left real estate worth r dollars and personal property worth p dollars, the property as a whole to be divided equally among the children. Find a formula for l, the property inherited by each child.

Illustrate the use of this formula by a numerical example composed by yourself.

10. Express the relation between divisor, dividend, quotient, and remainder, using d, D, q, and r to represent these quantities respectively.

11. Choosing your own letters, by use of them express the relation between minuend, subtrahend, and remainder.

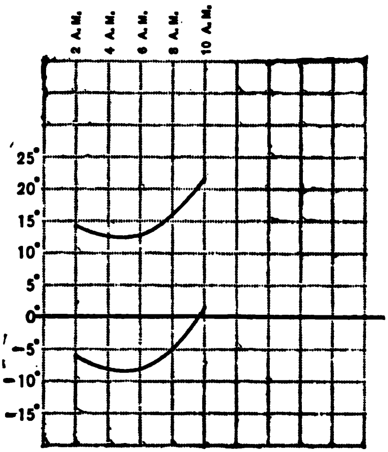
12. In making coffee, the rule is to allow one tablespoonful for each person, and one for the pot. Formulate this rule, using suitable letters.

EXERCISE 15

1. On Feb. 1, 1919, and Feb. 1, 1920, at the same place, the temperatures at two-hour intervals were as follows:

	2 A.M.	4 A.M.	6 A.M.	8 A.M.	10 A.M.	NOON
Feb. 1, 1919...........	14°	12°	13°	16°	22°	24°
Feb. 1, 1920...........	−6°	−8°	−8°	−5°	1°	7°

	2 P.M.	4 P.M.	6 P.M.	8 P.M.	10 P.M.	MID-NIGHT
Feb. 1, 1919...........	25°	23°	19°	15°	10°	8°
Feb. 1, 1920...........	12°	10°	7°	4°	1°	−3°



Graph the two sets of temperatures on the same diagram.

The beginning of the diagram is shown at the right. Let the pupil complete the two graphs.

In the construction of the following graphs, let the pupil determine whether each graph should be a smooth curve or a broken line.

2. At the same place on Jan. 15, 1919, and Jan. 15, 1920, the temperatures at three-hour intervals were as follows:

	3 A.M.	6 A.M.	9 A.M.	NOON	3 P.M.	6 P.M.	9 P.M.	MID-NIGHT
Jan. 15, 1919.....	9°	12°	15°	17°	18°	15°	14°	12°
Jan. 15, 1920.....	−5°	−7°	−2°	3°	7°	5°	2°	−3°

Graph these two sets of temperatures on the same diagram. From the graph determine on which of the two days the difference between the highest and lowest temperatures was the greatest.

3. At a given place on Aug. 4, 1919, and Aug. 4, 1920, the temperatures at one-hour intervals were as follows:

	7 A.M.	8 A.M.	9 A.M.	10 A.M.	11 A.M.	Noon	1 P.M.	2 P.M.	3 P.M.	4 P.M.	5 P.M.	6 P.M.
Aug. 4, 1919.......	62°	66°	70°	76°	80°	86°	92°	85°	74°	86°	87°	81°
Aug. 4, 1920.......	59°	63°	67°	72°	77°	85°	89°	91°	87°	84°	78°	74°

Graph these two sets of temperatures on the same diagram.

On one of the two days a thunderstorm occurred in the afternoon. From an examination of the two graphs, determine on which of the two days the storm occurred. At what time on the two days were the temperatures the same?

4. The following were the mean or average temperatures at New York and San Francisco for a long period of years:

	Jan. 1	Feb. 1	Mar. 1	April 1	May 1	June 1	July 1	Aug. 1	Sept. 1	Oct. 1	Nov. 1	Dec. 1
New York.........	31°	31°	35°	42°	54°	64°	71°	73°	69°	61°	49°	39°
San Francisco......	50°	52°	54°	55°	57°	58°	58°	59°	60°	59°	51°	56°

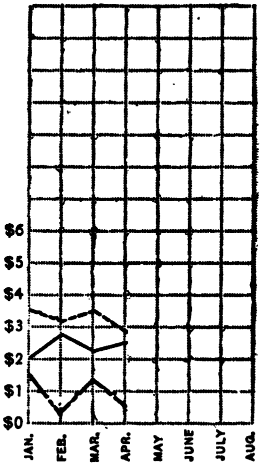
From the graphs of these two sets of temperatures, determine as closely as you can on what dates the mean temperatures of the two places are the same.

Which of the two cities has the more uniform temperature? What is the extreme range of temperature in each of the two cities?

5. A boy's savings in the months of two successive years were as follows:

	JAN.	FEB.	MAR.	APRIL	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
First year	$2.00	$2.75	$2.10	$2.42	$2.80	$3.12	$4.25	$4.17	$3.76	$4.15	$3.89	$2.78
Second year	3.50	3.10	3.40	2.93	3.15	3.75	8.90	12.40	6.21	5.20	4.15	3.17

Graph these two sets of numbers on the same diagram. Also on this diagram, construct a third graph showing the increase of savings in the second year over the first year. The beginnings of the three graphs are shown on the diagram at the right.



In the summer of the second year the boy worked on a farm. How is the result shown on the diagram?

6. For the different months of a certain year, the expenditures and receipts of a certain business were as follows in dollars:

	JAN.	FEB.	MAR.	APRIL	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
Expenditures	$810	$790	$820	$872	$850	$890	$920	$950	$921	$874	$882	$923
Receipts	1024	1170	1220	1380	1460	1570	1320	1603	1727	1629	1548	1276
Net Profits	214	380										

Let the pupil fill in the vacant spaces.

In constructing graphs of these three sets of numbers (on the same diagram), let each space on the vertical axis represent $200.

A convenient method of determining the vertical axis scale for numbers like the above is the following. If, for example, only 12 spaces are available on the vertical axis, divide by 12 the largest number to be represented, and take some convenient number larger than the quotient as the number represented by one space. Thus, in this example: $1727 \div 12 = 144^-$. Hence, let one space represent $200.

7. The following table gives (in number of thousands) the population of New York and Chicago at ten-year intervals:

YEAR	NEW YORK	CHICAGO	YEAR	NEW YORK	CHICAGO
1800	60		1870	942	299
1810	96		1880	1165	503
1820	124		1890	1441	1100
1830	202		1900	3437	1699
1840	313		1910	4767	2185
1850	516	30	1920		
1860	814	109	1930		

Graph these numbers on the same diagram. Let each space on the vertical axis represent 200,000 population.

How do you account for the sudden increase in the population of New York between 1890 and 1900?

8. The following table gives the amount of $1 at simple interest at 6%, at compound interest at $3\frac{1}{2}\%$, and also at compound interest at 6%, for 5, 10, 15, etc., years. Graph these results on the same diagram.

From the diagram, determine as exactly as you can, the number of years when the amount of $1 at $3\frac{1}{2}\%$ compound interest equals that of $1 at 6% simple interest.

Also when the amount at 6% compound interest is double the amount at simple interest.

Years	0	5	10	15	20	25	30	35
Amount at 6% Simple Interest......	$1	$1.30	$1.60	$1.90	$2.20	$2.50	$2.80	$3.10
Amt. at 3½% Compound Interest...	$1	1.19	1.41	1.68	1.99	2.36	2.81	3.33
Amt. at 6% Compound Interest...	1	1.34	1.79	2.40	3.21	4.29	5.74	7.66

9. The adjoining table gives in millions of dollars the exports and imports of the United States for the years specified. Make a graph of the exports and also of the imports on the same diagram. (Let each space on the vertical axis stand for 500 millions of dollars; let each of the first three spaces on the horizontal axis stand for 5 years, and each space after that for 1 year.)

Year	Exports	Imports	Year	Exports	Imports
1890	857	823	1911	2092	1532
1895	825	802	1912	2399	1818
1900	1478	829	1913	2484	1793
1905	1627	1179	1914	2114	1789
1906	1798	1321	1915	3547	1779
1907	1923	1423	1916	5482	2392
1908	1753	1116	1917	6228	2953
1909	1728	1476	1918	6125	3050
1910	1866	1563	1919	7022	3904

Also on the same diagram, construct another curve showing the excess of exports over imports. How do you account for the great increase of exports over imports during the years 1914–1919?

EXERCISE 16

Review

1. Add $7a^2-3ab$, $5a^2$, $3ab-7b^2$, $16a^2-b^2-7ab$.

2. Simplify $5.8p^2-[8.2p^2-(p^2-.4)+.06]$.

3. One number exceeds another number by 12, and three times the smaller number exceeds twice the larger number by 18. Find the number.

4. Find a formula for the number (B) of bricks, each $a'' \times b''$, required in making a pavement l feet long and w feet wide, if 5 per cent be added as an allowance for breakage.

5. The populations (in hundred thousands) of Illinois, New York, and Virginia at ten-year intervals are given in the following table:

YEAR	1800	1810	1820	1830	1840	1850	1860	1870	1880	1890	1900	1910
Illinois	....	....	.5	1.6	4.8	8.5	17.1	25.3	30.8	38.3	48.2	56.4
New York	6.9	9.6	13.7	19.2	24.3	31.0	38.8	43.8	50.8	60.0	72.7	91.1
Virginia	8.8	9.7	10.7	12.1	12.4	14.2	16.0	12.3	15.1	16.6	18.5	20.6

Graph these facts as three curves on one diagram.

From the graphs determine as accurately as you can the year in which the population of New York State equaled that of Virginia. Also that in which the population of Illinois equaled that of Virginia.

6. Solve and check $5[2(x+1)-(x+3)]-3\{x+2(6x-15)\}=0$.

7. Subtract the product of $-5a$ and $4a$ from the product of $-6a$ and $-8a$.

8. Add the numbers 496, 508, 492, 501, 509, 492. Then add the set of numbers equivalent to these as tabulated at the right.

Which method of getting the required sum is easier? How is the efficiency value of negative number illustrated in this comparison?

$$\begin{array}{r} 500-4 \\ 500+8 \\ 500-8 \\ 500+1 \\ 500+9 \\ \underline{500-8} \end{array}$$

(This process is also interesting since it was in connection with it that the minus and plus signs were invented. This occurred in the city of Nuremberg, Germany, near the year 1489.)

9. The following tabulation gives (in thousands of dollars) the monthly revenues and expenses of a certain business during a certain year.

Month	Jan.	Feb.	Mar.	April	May	June
Revenue..............	18	21	19	23	24	20
Expenses.............	12	13	10	11	18	21
Net Income...........						

Month	July	Aug.	Sept.	Oct.	Nov.	Dec.
Revenue..............	18	19	24	27	31	32
Expenses.............	24	20	17	11	13	14
Net Income...........						

Let the pupil fill in the figures for net income and then make a graph of the three sets of figures on one diagram.

10. Explain the difference between an identity and an equation. Give an example of each. Also state which of the following are indentities and which equations:

(1) $(x+4)(x-4)=x^2-16.$ (3) $(x-3)(x+5)=x^2+2x-15.$
(2) $(x+4)(x-4)=x^2-2x.$ (4) $(x-3)(x+5)=x^2.$
(5) $(a+b)(a-b)=a^2-b^2.$

11. Two cars start 200 miles apart and move toward each other at the rate of a and b miles per hour, respectively. Find a formula for the number of miles (M) they will be apart at the end of t hours.

Also find this formula if the cars start d miles apart.

12. A man walked 15 miles, rode a certain distance, and then took a boat for twice as far as he had previously traveled. Altogether he went 120 miles. How far did he go by boat?

13. Multiply $3x^2-5.2x+6$ by $.4x-.5$.

14. At a certain distance (a feet) below the earth's surface, the temperature of a coal mine was 78°. After that the temperature increased 1° for every 50 ft. of increase in the depth of the mine. Obtain a formula for the temperature (T) at a depth of d feet.

15. Divide $2\,x^{2n+3}-9\,x^{2n+2}+11\,x^{2n+1}-3\,x^{2n}$ by $2\,x^{n-1}-3\,x^{n-2}$

16. Subtract $px^2+qxy-ry^2$ from $ax^2-3\,y^2$.

17. Separate 4800 into three parts, such that the second part is three times the first, and one-half of the third part exceeds the second by 400.

18. After proceeding a certain distance, a steamer had a tons of coal left and was using b tons of coal a day. Find a formula for T, the number of tons of coal left in the steamer d days later.

In this formula find the value of T when $a=842$, $b=62$, and $d=4$. Also when $d=-4$. What is the meaning of this last result?

19. In $L=\frac{nt^2w}{d}$, find the value of L when $n=60$, $t=3.4$, $d=42$, and $w=623$. (Use cancellation.)

20. Copy the following tabulation and fill in the vacant places with the needed rule or formula:

SUBJECT	BRIEF RULE	FORMULA
Percentage		$b=\frac{p}{r}$
Area of triangle		$A=\frac{1}{2}\,bh$
Circle	Diameter = circumference ÷ π	
Area of trapezoid		$A=\frac{1}{2}\,(b+b')h$
Motion of body	Distance = rate × time	
Cubic yd. in cellar		$Y=\frac{lwh}{27}$
Circle	Area of circle = π times radius squared	
Interest	Rate = (interest) ÷ (principal × time)	

CHAPTER II

ABBREVIATED MULTIPLICATION AND DIVISION; FACTORING

ABBREVIATED MULTIPLICATION AND DIVISION

6. Cases I, II, and III of Short Multiplication.

I. $(a+b)^2=a^2+2\,ab+b^2$.

Ex. $(4\,x^2+3\,y)^2=16\,x^4+24\,x^2y+9\,y^2$. *Product.*

II. $(a-b)^2=a^2-2\,ab+b^2$.

Ex. $(3\,x^{n-1}-2\,xy^{n+1})^2=9\,x^{2n-2}-12\,x^ny^{n+1}+4\,x^2y^{2n+2}$. *Product.*

III. $(a+b)(a-b)=a^2-b^2$.

Ex. $$\begin{aligned}(x+y-2\,z)(x-y+2\,z)&=[x+(y-2\,z)]\,[x-(y-2\,z)]\\&=x^2-(y-2\,z)^2\\&=x^2-(y^2-4\,yz+4\,z^2)\\&=x^2-y^2+4\,yz-4\,z^2.\quad \textit{Product.}\end{aligned}$$

EXERCISE 17

At sight write the product of

1. $(3\,x+4\,a)^2$.
2. $(3\,x-4\,a)^2$.
3. $(\frac{2}{3}\,p-\frac{1}{2}\,q)^2$.
4. $(10\,xy^3+1)^2$.
5. $[(a+x)-5]^2$.
6. $(x-a-b)^2$.
7. $(3\,x^n-5\,y^{n-1})^2$.
8. $(3x-2\,y+3\,z)^2$.
9. $(.5\,a-.04\,b^2)^2$.
10. $[(a+b)-(x+y)]^2$.
11. 103^2.
12. 99.8^2.
13. $(x^3+9)(x^3-9)$.
14. $(.4\,x-.05)(.4\,x+.05)$
15. $(a+b+2\,x)(a+b-2\,x)$.
16. $(2\,x^n+5\,y^m)(2\,x^n-5\,y^m)$.
17. $(a^{n+1}+\frac{1}{2}\,b^{n-1})(a^{n+1}-\frac{1}{2}\,b^{n-1})$.
18. 997×1003.
19. 96×104.
20. 9.7×10.3.
21. $[(2\,x-1)+y][(2\,x-1)-y]$.

22. $[4+(x+p)][4-(x+p)]$.

23. $(4+x+p)(4-x-p)$.

24. $(2x+3y-5)(2x-3y+5)$.

25. $(x^2+xy+y^2)(x^2-xy+y^2)$.

26. Multiply $a+2b-3x$ by $a+2b+3x$ by long multiplication. Also by short multiplication. Compare the amount of work in the two processes.

7. Cases IV and V of Short Multiplication.

IV. $(x+a)(x+b)=x^2+(a+b)x+ab$

Ex. $(x+3y)(x-2y)=x^2+xy-6y^2$. *Product.*

V. $(ax+b)(cx+d)=acx^2+(ad+bc)x+bd$.

Ex. $(3x+5)(2x-7)=6x^2-11x-35$. *Product.*

EXERCISE 18

At sight write the product of

1. $(x+3)(x+5)$.

2. $(p^2+.4)(p^2-.1)$.

3. $(p^2q^2-5a)(p^2q^2+3a)$.

4. $(y-a)(y+b)$.

5. $(x-7)(3x+4)$.

6. $(2x^2-3y^3)(4x^2-7y^3)$.

7. $(2x-.3)(4x-.5)$.

8. $(7x^3-3xy^2)^2$.

9. $(a^2+3x)(a^2-5x)$.

10. $(x^{a+b}-x^{a-b})^2$.

11. $(x^2+2xy^{n-1})^2$.

12. $(y+\frac{2}{3})(y-\frac{1}{3})$.

13. $[(a+b)+7][(a+b)-3]$.

14. $(a+.2b)(2a-.3b)$.

15. $(\frac{1}{2}x+\frac{3}{2}a)(\frac{3}{2}x-\frac{1}{2}a)$.

16. $(2x-.3)(2x+.3)$.

17. $(3ax^n-2a^{n-1})^2$.

18. $(-2x+3y)^2$.

19. $(-a-b)(-a+b)$.

20. $(x+y+5)(x+y-5)$.

21. $(x+y+5)^2$.

22. $(x+y+5)(x+y-3)$.

23. $(4a^2+2ax+x^2)(4a^2-2ax+x^2)$.

24. $(9a^2+3a+1)(9a^2-3a+1)$.

25. 995^2.

26. 995×1005.

27. 100.3×99.7.

Simplify the following, using methods of abbreviated multiplication as far as possible:

28. $(x+y)^2+(x-y)^2$. **29.** $(3x-y)^2-(3x+y)^2$.

30. $(3x-1)^2-(3x+2)(3x-2)$.

31. $(5a-3b)(2a-b)-3(a-5b)^2+(a-5b)(a+4b)$.

32. $(a+b+c)(a+b-c)-3(a-c)^2-(b+c)^2$.

33. Why is it that the value of $(2x-3y)^2$ is the same as that of $(3y-2x)^2$? Make up two similar expressions that have this property.

Compute in a short way:

34. The area of a field 104 rd. long and 96 rd. wide.

35. The cost of 51 yd. of cloth at 49 cents a yard.

8. Division of Sum or Difference of Two Cubes.

By actual division we can obtain,

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2, \quad \text{and} \quad \frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

Hence, in general language,

The **sum** *of the cubes of two quantities is divisible by the sum of the quantities, and the quotient is the square of the first quantity, minus the product of the two quantities, plus the square of the second quantity;* also

The **difference** *of the cubes of two quantities is divisible by the difference of the quantities, and the quotient is the square of the first quantity, plus the product of the two quantities, plus the square of the second.*

Ex. 1.
$$\frac{8x^3-27y^3}{2x-3y}=\frac{(2x)^3-(3y)^3}{2x-3y}$$
$$=(2x)^2+(2x)(3y)+(3y)^2.$$
$$=4x^2+6xy+9y^2. \quad \textit{Quotient.}$$

Ex. 2. $\frac{(a-b)^3+27}{(a-b)+3}=(a-b)^2-3(a-b)+9$

$=a^2-2ab+b^2-3a+3b+9.$ *Quotient.*

Let the pupil check the work in these examples.

EXERCISE 19

Write at sight the quotient for each of the following and check each result:

1. $\frac{a^3+8}{a+2}.$
2. $\frac{x^3-1}{x-1}.$
3. $\frac{27x^3-64}{3x-4}.$
4. $\frac{1+8x^6}{1+2x^2}.$
5. $\frac{125-x^9}{5-x^3}.$
6. $\frac{27a^6+y^{12}}{3a^2+y^4}.$
7. $\frac{x^6+y^6}{x^2+y^2}.$
8. $\frac{.008x^3-y^3}{.2x-y}.$
9. $\frac{\frac{8}{27}a^3+\frac{1}{64}b^6}{\frac{2}{3}a+\frac{1}{4}b^2}.$

10. Divide $8a^3+27b^3$ by $2a+3b$ by the method of long division. Now write out the same quotient by the method of § 8, p. 29. Estimate how much of the labor of division is saved by using the second method of obtaining the quotient.

Obtain in the short way the quotient for each of the following:

11. $\frac{c^3+(1-x)^3}{c+(1-x)}.$
12. $\frac{8-(x+y)^3}{2-(x+y)}.$
13. $\frac{27x^6+125y^9}{3x^2+5y^3}.$
14. $\frac{(a-1)^3-x^6}{(a-1)-x^2}.$

Write the binomial divisor and the quotient for

15. $\frac{8a^3-x^3}{\quad}=$
16. $\frac{8a^3-27x^3}{\quad}=$
17. $\frac{1-64x^3}{\quad}=$
18. $\frac{8a^3+1}{\quad}=$
19. $\frac{a^6+y^6}{\quad}=$
20. $\frac{x^{12}+y^{12}}{\quad}=$

21. Find the difference in value between x^3+y^3 and $(x+y)^3$, when $x=2$ and $y=3$.

9. Division of Sum or Difference of any Two Like Powers.

By actual division we can obtain,

$$\frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3. \quad \textit{Quotient.}$$

$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3. \quad \textit{Quotient.}$$

a^4+b^4 is not divisible by either $a+b$ or $a-b$. But

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4. \quad \textit{Quotient.}$$

$$\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4. \quad \textit{Quotient.}$$

The **difference** of two like **even** powers of two quantities is divisible by the **sum** of the quantities, and also by their **difference**;

The **sum** *of two like* **odd** *powers of two quantities is divisible by the* **sum** *of the quantities;*

The **difference** *of two like* **odd** *powers of two quantities is divisible by the* **difference** *of the quantities.*

For the quotient in all these cases—

(1) The number of terms in a quotient equals the degree of the powers whose sum or difference is divided;

(2) The terms of each quotient are homogeneous (since the exponent of a decreases by 1 in each term, and that of b increases by 1 in each term).

(3) *If the divisor is a difference, the signs of the quotient are all plus; if the divisor is a sum, the signs of the quotient are alternately plus and minus.*

In the above statements the parts in *italics* should be committed to memory.

Ex. 1. $\frac{a^7+x^7}{a+x}=a^6-a^5x+a^4x^2-a^3x^3+a^2x^4-ax^5+x^6.$

Quotient.

Ex. 2. $\frac{32x^5+y^5}{2x+y}=\frac{(2x)+y^5}{2x+y}$

$$=(2x)^4-(2x)^3y+(2x)^2y^2-(2x)y^3+y^4$$
$$=16x^4-8x^3y+4x^2y^2-2xy^3+y^4.$$

Quotient.

EXERCISE 20

Write the quotient at sight and check each result:

1. $\frac{a^5+x^5}{a+x}$.
2. $\frac{a^5-x^5}{a-x}$.
3. $\frac{b^7+y^7}{b+y}$.
4. $\frac{a^7-128}{a-2}$.
5. $\frac{x^5-1}{x-1}$.
6. $\frac{32x^5-y^5}{2x-y}$.
7. $\frac{a^{11}+x^{11}}{a+x}$.
8. $\frac{x^{10}-y^{15}}{x^2-y^3}$.
9. $\frac{243-a^5}{3-a}$.

10. Divide $32a^5+x^5$ by $2a+x$ by the method of long division. Now write the same quotient by the method of § 9, p. 31. Estimate how much of the labor of division is saved by using the second method.

11. $\frac{a^5+y^5}{\quad}=$
12. $\frac{a^5-y^5}{\quad}=$
13. $\frac{b^7-x^7}{\quad}=$
14. $\frac{x^5-32}{\quad}=$
15. $\frac{a^7+128}{\quad}=$
16. $\frac{y^5+1}{\quad}=$

17. Divide a^5+b^5 by $a+b$. Also $(a+b)^5$ by $a+b$.

18. Find the difference in value between x^5-y^5 and $(x-y)^5$, when $x=3$ and $y=2$.

FACTORING

10. Cases I and II of Factoring.

CASE I. **A Polynomial having a Common Factor in all its Terms.**

Ex. $6x^2y-9x^3y^2-18ax^4-12bx^2y$

$$=3x^2(2y-3xy^2-6ax^2-4by).$$ *Factors.*

CASE II. A Trinomial that is a Perfect Square.

Ex. 1. $16x^2-24xy+9y^2=(4x-3y)^2$. *Factors.*

Ex. 2. $a^2-4ab+4b^2-9x^2=(a-2b)^2-9x^2$

$=(a-2b+3x)(a-2b-3x)$. *Factors.*

EXERCISE 21

Factor and check:

1. $3x^2+6x^3+9x^4$.
2. $15ax+10ay-5az$.
3. $\frac{2}{5}x^3-\frac{2}{5}x^2-\frac{2}{5}x$.
4. $2\pi r^2+2\pi rh$.
5. $x^2+4ax+4a^2$.
6. $x^3+6x^2y+9xy^2$.
7. $4a^{3n}-8a^{2n}+a^{4n}$.
8. $.49+1.4b+b^2$.
9. $x^{2n}+2x^ny^n+y^{2n}$.
10. $9a^2+4ab+\frac{4}{9}b^2$.
11. $(a+x)^2+2b(a+x)+b^2$.
12. $(x+y)^2-2ax-2ay+a^2$.
13. $x^2+y^2+z^2+2xy+2xz+2yz$.

Compute in the shortest way:

14. 73×\$1.25+27×\$1.25.
15. 186×\$5.46+27×\$5.46−113×\$5.46.

11. Cases III and IV of Factoring.

CASE III. The Difference of Two Squares.

Ex. 1. $16x^4-9y^6=(4x^2+3y^3)(4x^2-3y^3)$. *Factors.*

Ex. 2. $4a^2-x^2+2xy-y^2=4a^2-(x^2-2xy+y^2)$

$=4a^2-(x-y)^2$

$=(2a+x-y)(2a-x+y)$. *Factors.*

Ex. 3. Compute the value of $82.1^2-7.9^2$ in the shortest way.

$82.1^2-7.9^2=(82.1+7.9)(82.1-7.9)$

$=90\times 74.2$

$=6678$. *Ans.*

Case IV. A Trinomial of the Form x^2+bx+c.

Ex. $x^2-5\,xy-14\,y^2=(x-7\,y)(x+2\,y)$. *Factors.*

EXERCISE 22

Factor and check:

1. $25-9x^2$.
2. $16\,a^2x-b^2x$.
3. $144-.0009\,x^2$.
4. $.36\,x^3-81\,xy^2$.
5. $\frac{9}{4}\,p^3-\frac{1}{16}\,pq^2$.
6. by^8-b.
7. $x^{2n}-y^{4n}$.
8. $x^2-(a+b)^2$.
9. $x^2-a^2-2\,ab-b^2$.
10. $4\,x^2-p^2+2\,pq-q^2$.
11. $4\,a^2-16\,b^2+4\,a+1$.
12. $9-16\,a^2-8\,a-1$.
13. $\frac{1}{16}-(x-3\,y)^2$.
14. $(a+b)^2-9(a+b-1)^2$.
15. $9(x+y)^2-4(2\,x-3\,y)^2$.

Compute in a short way by aid of factoring:

16. 476^2-324^2.
17. $89.73^2-85.27^2$.
18. Evaluate $\pi R^2-\pi r^2$ when $\pi=\frac{22}{7}$, $R=32$, and $r=24$.

Factor and check:

19. $x^2-7\,x+12$.
20. $p^{2n}+3\,p^n-70$.
21. $b^4+3\,b^2-28$.
22. $a^2+4\,a-165$.
23. $y^{2n-2}-.5\,y^{n-1}+.06$.
24. $a^2b^2-9\,ab+18$.
25. $x^6-13\,x^4+12\,x^2$.
26. $x^2-(p+q)x+pq$.
27. $(x+y)^2-5(x+y)+6$.
28. $(2\,x-3)^2+3(2\,x-3)-10$.
29. $a^3-13\,a^2+36\,a$.
30. $a^3-12\,a^2+36\,a$.
31. $1-16x^4$.
32. $a^2-x^2-4\,xy-4\,y^2$.
33. $y^2+(a+b)y+ab$.
34. $2\,ab-a^2-b^2+y^2$.
35. $ap^4-.0016\,a^5$.
36. $(x+y)^2-6(x+y)+8$.
37. $(x+y)^2-6(x+y)+9$.
38. $x^{2n}-2\,x^ny^2+y^4$.

12. Case V. A Trinomial of the Form ax^2+bx+c.

The essential part of the process of factoring a trinomial of the form ax^2+bx+c lies in determining two factors of the first term and two factors of the last term, such that the algebraic sum of the cross products of these factors equals the middle term of the trinomial.

Ex. 1. Factor $10x^2+13x-3$.

The possible factors of the first term are $10x$ and x, $5x$ and $2x$. The possible factors of the third term are -3 and 1, 3 and -1. In order to determine which of these pairs will give $+13x$ as the sum of their cross products, it is convenient to arrange the pairs thus:

$$\begin{array}{ccc} 10x, & & -3 \\ & \times & \\ x, & & 1 \end{array} \quad ; \quad \begin{array}{ccc} 5x, & & -1 \\ & \times & \\ 2x, & & 3 \end{array}$$

Variations may be made mentally by transferring the minus sign from 3 to 1; and also by interchanging the 3 and the 1.

It is found that the sum of the cross products of

$$\begin{array}{cc} 5x, & -1 \\ 2x, & 3 \end{array} \quad \text{is } +13x$$

Hence, $10x^2+13x-3=(5x-1)(2x+3)$. *Factors.*

Let the pupil check the work.

Hence, in general, to factor a trinomial of the form

$$ax^2+bx+c,$$

Separate the first term into two such factors, and the third term into two such factors, that the sum of their cross products equals the middle term of the trinomial;

As arranged for cross multiplication, the upper pair taken together and the lower pair taken together form the two factors.

If, in a given example, the number which takes the place of either a or c has many factors, the method of factoring used in the following example is a saving of labor.

Ex. 2. Factor $20x^2+49x-48$.

Multiply and divide the given expression by 20, the coefficient of the first sum.

$$\text{Hence,}\quad 20x^2+49x-48=\frac{(20x)^2+49(20x)-960}{20}$$

$$=\frac{(20x+64)(20x-15)}{20}$$

$$=\frac{4(5x+16)5(4x-3)}{20}$$

$$=(5x+16)(4x-3).\quad \textit{Factors.}$$

EXERCISE 23

Factor and check:

1. $2a^2+3a+1$.
2. $3a^2-4a+1$.
3. $2x^2-5x+3$.
4. $3p^2+8p+5$.
5. $3p^2+16p+5$.
6. $7a^2-2ab-5b^2$
7. $3x^4-11x^3+6x^2$.
8. $10x^2-13xy-3y^2$.
9. $12a^2-7ab-12b^2$.
10. $9a^4-13a^2+4$.
11. $3x^2-1.4x+.08$.
12. $6x^2y-2xy-4y$.
13. $9a^4-148a^2+64$.
14. $12-7x-12x^2$.
15. $2x^{2n}+5x^n-3$.
16. $3(x-y)^2+7(x-y)z-6z^2$.
17. $3(x^2+2x)^2-5(x^2+2x)-12$.
18. $px^2+(p-q)x-q$.

13. Case VI. Sum or Difference of Two Like Odd Powers.

From § 8, p. 29, $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2$.

Hence, $a^3+b^3=(a+b)(a^2-ab+b^2)$. . . . (1)

In like manner, $a^3-b^3=(a-b)(a^2+ab+b^2)$. . . . (2)

But any algebraic expressions may be used instead of a and b in formulas (1) and (2).

Ex. 1. Factor $27x^3-8y^3$.

$$27x^3-8y^3=(3x)^3-(2y)^3.$$

Use $3x$ for a and $2y$ for b in (2) above.

$$27x^3-8y^3=(3x-2y)(9x^2+6xy+4y^2).\quad \textit{Factors.}$$

In working examples of this type, it is often convenient to call $3x-2y$ the "divisor factor" and $9x^2+6xy+4y^2$ the "quotient factor." Why are these names appropriate in this case?

Ex. 2. Factor a^6+8b^9.

$$\begin{aligned}a^6+8b^9&=(a^2)^3+(2b^3)^3\\&=(a^2+2b^3)(a^4-2a^2b^3+4b^6).\quad \textit{Factors.}\end{aligned}$$

Ex. 3. Factor $(a+b)^3-x^3$.

$$(a+b)^3-x^3=[(a+b)-x]\,[(a+b)^2+(a+b)x+x^2].$$

Let the pupil check the above examples.

In general, to factor the sum or difference of two cubes,

Obtain the values of a and b in the given example, and substitute these values in formula (1) *or* (2).

By the use of § 9 (p. 31), the sum or difference of two like odd powers higher than the third may be factored.

Ex. 4. Factor a^5-b^5.

$$a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4).$$

Ex. 5. Factor x^5+32y^5.

$$\begin{aligned}x^5+32y^5&=x^5+(2y)^5\\&=(x+2y)[x^4-x^3(2y)+x^2(2y)^2-x(2y)^3+(2y)^4]\\&=(x+2y)(x^4-2x^3y+4x^2y^2-8xy^3+16y^4).\quad \textit{Factors.}\end{aligned}$$

The sum of certain two like even powers may be factored by regarding the expression as the sum of two like odd powers.

Ex. 6. Factor a^6+b^6.

$$\begin{aligned}a^6+b^6&=(a^2)^3+(b^2)^3\\&=(a^2+b^2)(a^4-a^2b^2+b^4).\quad \textit{Factors.}\end{aligned}$$

But a^2+b^2, a^4+b^4, a^8+b^8 cannot be factored by any elementary method and are therefore prime expressions.

EXERCISE 24

Factor and check:

1. m^3-n^3.
2. c^3+8d^3.
3. $27-x^3$.
4. $a^3+8b^3c^3$.
5. x^3-125.
6. $64y^3-27$.
7. a^3b^3+1.
8. $1-1000x^3$.
9. $27x^4+a^3x$.
10. $512x^3-y^6$.
11. $a+343a^4$.
12. a^6-x^6.
13. $x^{12}-y^6$.
14. a^6-64n^{12}.
15. $250x-2x^7$.
16. $8x^6+y^3$.
17. $16x^4y^6-54xz^3$.
18. x^5+y^5.
19. x^7-y^7.
20. a^6+m^6.
21. $x^{12}+y^{12}$.
22. a^7-128b^7.
23. $a^{11}+x^{11}$.
24. a^9+b^9.
25. $32x^5-1$.
26. $a^{11}-b^{11}$.
27. $243-x^5$.
28. $a^{10}-b^{10}$.
29. $a^{10}+b^{10}$.
30. $32x^5-a^{10}$.
31. a^6+y^9.
32. $8x^{12}+y^9$.
33. $.008x^3-y^3$.

34. Make up a binomial expression whose terms contain unlike exponents and which can be factored as the sum of two cubes. Also one that can be factored as the sum of two 5th powers.

35. Make up a binomial the exponents in whose terms are even numbers, and which can be factored as the sum of two cubes. Also one that can be factored as the sum of two 5th powers.

36. State which of these expressions can be factored:

x^3+y^8
x^3+y^9
x^3+y^6
x^3+y^5
x^3-y^{12}
x^6-y^9
x^3-y^{10}
x^6+y^8
x^6+y^9
x^6+y^{10}
x^6+y^{12}
x^5-y^{10}

Factor and check each of these miscellaneous examples:

37. x^2-4a^2.
38. x^3-8a^3.
39. $x^2-4ax+4a^2$.
40. $x^2-4ax+3a^2$.
41. x^6+a^9.
42. x^6-a^6.
43. $x^{12}+y^9$.
44. $x^{12}-y^{12}$.

45. x^4-a^4.

46. x^5+a^5.

47. $x^2-4(a+b)^2$.

48. $6\,a^2-13\,a+6$.

49. $16\,x^2-8\,xy+y^2$.

50. $x^4+27\,a^3x$.

51. Make up and work an example in factoring to illustrate each case treated thus far.

14. Case VII. A Polynomial whose Terms may be Grouped so as to be Divisible by a Binomial Divisor.

Ex. 1.
$$\begin{aligned} ax-ay-bx+by &= (ax-ay)-(bx-by)\\ &= a(x-y)-b(x-y)\\ &= (a-b)(x-y). \quad \textit{Factors.} \end{aligned}$$

The last step in the preceding process is sometimes clearer to the pupil when written in the following form:

$$\begin{array}{ll} (x-y)\ \underline{)a(x-y)-b(x-y)} & = (x-y)(a-b). \quad \textit{Factors.}\\ \qquad\qquad\ a \qquad\quad -b & \end{array}$$

Ex. 2.
$$\begin{aligned} 1+15\,a^4-5\,a-3\,a^3 &= 1-3\,a^3-5\,a+15\,a^4\\ &= (1-3\,a^3)-5\,a(1-3\,a^3)\\ &= (1-3a^3)(1-5\,a). \quad \textit{Factors.} \end{aligned}$$

EXERCISE 25

Factor and check:

1. $a(x+y)+b(x+y)$.
2. $ax+ay+bx+by$.
3. $(a-b)p+(a-b)q$.
4. $ap-bp+aq-bq$.
5. $(p+q)x+py+qy$.
6. $3x(a-b)+5\,ay-5\,by$.
7. $\frac{1}{2}\,ax-\frac{1}{2}\,ay+\frac{1}{3}\,bx-\frac{1}{3}\,by$.
8. $x^2(x+1)+5(x+1)$.
9. $x^3+x^2+5\,x+5$.
10. z^3-z^2-z+1.
11. $ab-by-a+y$.
12. x^5-x^4-4x+4.
13. $a^2x^2-b^2x^2-a^2y^2+b^2y^2$.
14. a^3+b^3+a+b.
15. $a^3-b^3+a^2-b^2$.
16. $a^3-b^3-a^2+b^2$.
17. $a^3-b^3-(a-b)^2$.
18. $x-y+x^3-y^3$.

19. $x^2-ax+cx-ac.$

20. $5xy-10y-3x+6.$

21. $y^3+y^2+y+1.$

22. $ax^2-2a^2x-x+2a.$

23. $4a^2-a^2x^2+x^2-4.$

24. $x^3-1+2(x^2-1).$

25. $x(x+4)^2+4(x+4).$

26. $2(x^2-y^2)-x+y.$

Factor and check each of the following miscellaneous examples:

27. $a^3-8x^3.$

28. $a^2-16x^2.$

29. $ax-bx+ay-by.$

30. $a^2-a-2.$

31. $x^3-a^3+2(x-a).$

32. $3a^2-4a-4.$

33. $x^3-x^2+x-1.$

34. $x^6+27y^6.$

35. $a^8-y^8.$

36. $(a+b)^2-2(a+b)p+p^2.$

37. $(a+b)^2-(a+b)-2.$

38. $x^2+x^3-a^2-a^3.$

39. Make up and work an example to illustrate each case in factoring treated thus far.

15. Case VIII. Polynomials Factored by the Factor Theorem.

Ex. If 2 be substituted for x in x^3-7x+6, we find that x^3-7x+6 reduces to zero (since $8-14+6=0$).

We will now show, by a special method to be applied later, that x^3-7x+6 is exactly divisible by $x-2$, and hence, that $x-2$ is a factor of x^3-7x+6.

Dividing x^3-7x+6 by $x-2$, we may express the result thus:

$$x^3-7x+6=Q(x-2)+R, \quad . \; . \; . \; . \; . \quad (1)$$

where Q is the quotient and R stands for a possible remainder.

But if in identity (1) we let $x=2$, x^3-7x+6 becomes zero, and $Q(x-2)$ also $=0$ (since $x-2=0$). Hence, R must be zero.

Hence, $\qquad x^3-7x+6=Q(x-2),$

that is, $x-2$ is a factor of x^3-7x+6.

16. Factor Theorem. Similarly, in general, *if* E *is any polynomial expression consisting of powers of* x, *and* E *reduces to zero when* $x=a$, *then* $x-a$ *is a factor of* E.

For $E=Q(x-a)+R$, and R reduces to zero when $x=a$, in this identity.

Ex. 1. Determine whether $x-2$ is a factor of x^3+3x^2-4.

When $x=2$, the given expression reduces to $8+12-4$ or 16. Hence, by the factor theorem $x-2$ is not a factor of x^3+3x^2-4.

Ex. 2. Determine whether $x-1$ is a factor of x^3+3x^2-4.

On substituting 1 for x, the given expression reduces to $1+3-4$, or 0.

Hence, $x-1$ is a factor of x^3+3x^2-4.

Observe that a short way to determine whether an expression, E, is divisible by $x-1$, is to note whether the sum of the coefficients is equal to zero. Thus, $x^4+2x-13x^2-14x+24$ is divisible by $x-1$ since the sum of its coefficients, viz., $1+2-13-14+24$, equals 0.

Ex. 3. Factor $x^3-2x^2-9x+18$.

The last term of each factor of the given expression must be an exact divisor of 18. Hence, the only trial numbers which we need substitute for x are ±1, ±2, ±3, ±6, ±9. It is customary to try the smaller numbers first.

Substituting

1, we obtain $1-2-9+18$, or $7. \therefore x-1$ is not a factor;
-1, we obtain $-1-2+10+18$, or $25. \therefore x+1$ is not a factor;
2, we obtain, $8-8-18+18$, or $0. \therefore x-2$ is a factor.

Dividing $x^3-2x^2-9x+18$ by $x-2$, we obtain

$$\begin{aligned} x^3-2x^2-9x+18 &= (x^2-9)(x-2) \\ &= (x+3)(x-3)(x-2). \quad \textit{Factors.} \end{aligned}$$

EXERCISE 26

Oral

1. Is $x-1$ a factor of $x^3-13x+12$?
2. Is $x-2$ a factor of $x^3-8x+12$?
3. Is $x+2$ a factor of x^3+7x-6?
4. Is $x-3$ a factor of x^3+7x-6?
5. Is $x-1$ a factor of x^3-5x^2+7x-3?
6. Is $a-1$ a factor of $a^4+a^3-8a^2+8$?
7. Is $a-2$ a factor of $a^4+a^3-8a^2+8$?

EXERCISE 27

By use of the factor theorem, factor:

1. x^3-3x+2.
4. $x^3-11x^2+23x+35$.
2. x^3-3x-2.
5. $x^3-9x^2+26x-24$.
3. $x^4+2x^3-41x^2-42x+360$.
6. $x^3+x^2-14x-24$.
7. $x^4-4x^3-29x^2-144x-180$.
8. $x^4+5x^3-5x^2-45x-36$.
9. x^4-5x^2+4.
10. $x^3-3x^2-10x+24$.
11. $x^4+x^3-11x^2-9x+18$.

By use of the factor theorem, prove that

12. x^n-y^n is always divisible by $x-y$.
13. x^n+y^n is always divisible by $x+y$ when n is odd.
14. x^n+y^n is not divisible by $x+y$ when n is even.

EXERCISE 28

Review

Factor and check:

1. $x^4+4ax^3+4a^2x^2$.
3. $40a^3-5$.
2. $x^4-4a^2x^2$.
4. $.16x^4-.4x^3y+.25x^2y^2$.

5. $x^4-4ax^3+3a^2x^2$.
6. $3x^4-4ax^3-4a^2x^2$.
7. x^4-8a^3x.
8. $(a+x)y+az+xz$.
9. x^3-7x-6.
10. $x^3-.04x$.
11. $x^2-4(a+2y)^2$.
12. a^4-2a^2+1.
13. $\frac{3}{2}hx+\frac{3}{2}hy-\frac{3}{2}hz$.
14. $16x^{4n}-8x^{2n}+1$.
15. x^3-24x^2+144x.
16. $(2x+3y)^2-(5x-2y)^2$.
17. $.04x^2-.36y^2$.
18. $4x^5+5x^4y-6x^3y^2$.
19. $x^3-1.1x^2+.3x$.
20. $px-qx-3py+3qy$.
21. $p^2-x^2-2xy-y^2$.
22. $x^2+ax-ab-bx$.
23. $a^2-2ab+b^2-x^2$.
24. $9x^2-4a^2-4ab-b^2$.
25. $12a^2-4ab-3ax^2+bx^2$.
26. $12xy+25-4x^2-9y^2$.
27. $3x^7-3x$.
28. $1-5a^2+4a^4$.
29. $16y^4-.0081$.
30. $(m+3)^4-18(m+3)^2+32$.
31. $5ax^9-5a$.
32. $18x^3-3x^2-36x$.
33. $7a-7a^5b^4$.
34. $110-x-x^2$.
35. $(x^2+x-2)^2-(x^2-x+3)^2$.
36. $(x^2+y^2)^4-16x^4y^4$.
37. $x^{12}-y^{12}$.
38. $x^{12}+y^{12}$.
39. $x^3-6x^2+11x-6$.
40. $(5x-2)^2-2(5x-2)-15$.
41. $(a^2-6)^2-a^2$.
42. $32+n^5$.
43. m^7-n^7.
44. $(a+b)^2+4x^2+4(a+b)x$.
45. $x^4-10x^3+17x^2+40x-84$.
46. $16a^2b^2-(a-b)^4$.
47. $3(x+y)^2-2(x+y)-5$.
48. $a^3-y^3-a^2+y^2$.

49. Make a list of all the prime factors of x^8-1.

50. Make a list of all of the exact divisors of $x^4-x^3-6x^2$. How many of these are prime factors?

51. What is the smallest expression into which each of the following can be divided: $x-2$, $x-3$?

52. Answer the same for x^2-4, x^2-5x+6.

Find the least common multiple of the following:

53. $3x^2y$, $9xy^2$, $12xy$.

54. $6(x^3-y^3)$, $9(x^2-y^2)$.

55. $18(x-1)^2$, $24(x^2-1)$, $36x$.

56. $6a^2-12ab+6b^2$, $9a^2-9b^2$, $12(a-b)^3$.

57. $6(a^2-1)$, a^2-a, $4a^2-8a+4$.

58. $15ax^2(x-y)^2$, $3bxy(x^2-y^2)$, $9x(x+y)^3$.

59. Define least common multiple.

60. Write two expressions whose least common multiple is $12x^2(x-2)^2(x-1)$.

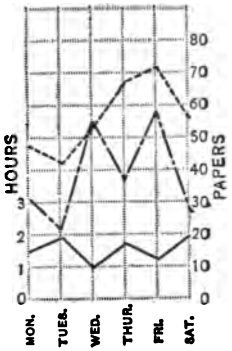
61. Write three fractions which have $24a^2b^3$ as their L. C. D.

62. Write three fractions whose L. C. D. is $9(a+b)(a-b)^2$.

EXERCISE 29

Ex. 1. A newsboy sold papers during the following numbers of hours on the successive days of a week: Monday, $1\frac{1}{2}$ hr.; Tuesday, 2 hr.; Wednesday, 1 hr.; Thursday, $1\frac{3}{4}$ hr.; Friday, $1\frac{1}{4}$ hr.; Saturday, 2 hr. The number of papers sold by him on these days were as follows: 47, 42, 54, 68, 72, 56. Graph these two sets of numbers on the same diagram.

Also on the same diagram, construct a curve showing the number of papers which the boy sold per hour.



This last curve is termed an *efficiency curve.*

SUG. It will be found convenient to tabulate the numbers to be graphed thus:

DAY OF WORK	MON.	TUES.	WED.	THURS.	FRI.	SAT.
No. hours	1½	2	1	1¾	1¼	2
No. papers sold	47	42	54	68	72	56
No. papers sold per hour	31	21	54	39	58	28

2. One week on successive workdays, a newsboy sold papers during the following number of hours: 1, 1½, 2½, 1¼, 1, 1½, 2½. The papers sold per day numbered as follows: 47, 72, 64, 76, 84, 92, 67. On the same diagram construct graphs of these two sets of facts. Also on this diagram, construct an efficiency curve.

3. The cost per month in dollars of running an auto truck for the first six months of a year and the number of miles run per month were as follows:

MONTH	JAN.	FEB.	MAR.	APRIL	MAY	JUNE
Cost	$31	$34	$43	$32	$38	$28
Miles run	632	476	462	610	705	800

Graph these two sets of facts on the same diagram. (For the first graph let one space on the vertical axis represent $10; and for the second graph, let one space represent 100 miles.)

Also on this diagram, construct an efficiency curve showing the number of miles run each month per dollar of expense. (In constructing the efficiency curve, let each space on the vertical axis represent 5 miles.)

4. For the first nine months of a given year, the sales and advertising expenses of a certain business were as follows:

MONTH	JAN.	FEB.	MARCH	APRIL	MAY
Sales	$7620	$8200	$7420	$7530	$8940
Ad. Expenses	225	240	316	320	460
Efficiency	34				

MONTH	JUNE	JULY	AUGUST	SEPTEMBER
Sales	$10,725	$12,010	$15,240	$17,903
Ad. Expenses	530	515	468	420
Efficiency				

Fill in the blank spaces and graph the three sets of facts on the same diagram. (Note that the efficiency curve shows the sales corresponding to each dollar spent in advertising.)

On May 1, a special advertising agent was employed. How is the effect shown on the efficiency curve?

5. Make up and work a similar example concerning money spent in advertising a magazine and the number of copies of the magazine sold.

6. The following table gives the normal or average height in inches and also weight of a boy at the ages specified. Graph these two sets of numbers on the same chart.

AGE IN YEARS	3	6	9	12	15	18	21
Height in inches	35	44	50	55.5	62.8	66.8	68.3
Weight in pounds	30	43	57	72	102	126	152
(Pounds) ÷ (inches)							

On this chart, also draw another curve showing the number of pounds in weight for each inch in height. From the chart, determine during what period the increase is most rapid in his height. In his weight. In his weight with respect to his height.

7. The following tabulation shows the growth in the population and in the value of the manufactures of a certain city for a certain period of years. Graph these facts on the same chart and add another curve showing the per capita value of the manufactures during the different years.

Year	1885	1890	1895	1900	1905	1910	1915	1920
Population in thousands..	32	36	42	46	51	53	58	76
M'f'res in millions.......	7.2	9.3	12.4	15.7	28.1	36.5	67.6	125.4
Per capita value of m'f'res.								

8. The following table gives for the years specified the population of the United States in millions and the area of United States in thousands of square miles.

Year	Pop. in Millions	Area in Thousands sq. mi.	Density per sq. mi.	Year	Pop. in Millions	Area in Thousands sq. mi.	Density per sq. mi.
1790	4	892		1860	31	3026	
1800	5	892		1870	39	3617	
1810	7	1720		1880	50	3617	
1820	10	1792		1890	63	3617	
1830	13	1792		1900	77	3743	
1840	17	1792		1910	101	3743	
1850	23	2997		1920			

Fill in the blank spaces and graph the three sets of numbers on one chart.

CHAPTER III

FRACTIONS; FRACTIONAL EQUATIONS

FRACTIONS

17. Important Properties of Fractions.

(1) *If the numerator and the denominator of a fraction are both multiplied or both divided by the same quantity, the value of the fraction is not changed.*

(2) *The signs of any* **even** *number of factors of the numerator and denominator of a fraction may be changed without changing the sign of the fraction;*

But if the signs of an **odd** *number of factors are changed, the sign of the fraction must be changed.*

Ex. 1. Reduce $\frac{x^2-4}{(2-x)^2}$ to its lowest terms.

$$\frac{x^2-4}{(2-x)^2}=\frac{(x+2)(x-2)}{(2-x)(2-x)}=\frac{(x+2)(x-2)}{(x-2)(x-2)}=\frac{x+2}{x-2}. \quad \textit{Ans.}$$

An example in fractions may be checked by letting each letter equal a number both in the original expression and in the answer. Thus in the preceding example, if we let $x=0$,

$$\frac{x^2-4}{(2-x)^2}=\frac{-4}{2^2}=-\frac{4}{4}=-1.$$

Also,
$$\frac{x+2}{x-2}=\frac{2}{-2}=-1.$$

Ex. 2. Reduce $\frac{x^3-x^2+x}{x^2+2}$ to a mixed number.

$$\begin{array}{l|l} x^3-x^2+\ x & x^2+2 \\ \underline{x^3\qquad +2x} & \overline{x\ -1} \\ \quad -x^2-\ x & \\ \quad \underline{-x^2\qquad -2} & \\ \qquad -\ x+2 & \end{array}$$

Hence, $$\frac{x^3-x^2+x}{x^2+2}=x-1-\frac{x-2}{x^2+2}.\quad \textit{Ans.}$$

Let the pupil check by letting $x=1$, both in the original example and in the answer.

EXERCISE 30

Reduce to its simplest form:

1. $\frac{27a^3y^2y^5}{36a^2x^4y^4}$.

2. $\frac{9xy-12y^2}{12x^2-16xy}$.

3. $\frac{18(a-b)^2}{45(a-b^2)^2}$.

4. $\frac{x^2-4}{x^2-4x+4}$.

5. $\frac{3x^2+4x-4}{x^2+3x+2}$.

6. $\frac{a^3-64}{a^2-16}$.

7. $\frac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}$.

8. $\frac{p^3-27}{9-6p+p^2}$.

9. $\frac{4-(x+y)^2}{(x-2)^2-y^2}$.

10. $\frac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2-2bc}$.

11. $\frac{6a^2-7ab+2b^2}{6a^2-ab-2b^2}$.

12. $\frac{(x-1)(2-x)(3-x)}{(1-x)(x-2)(x-4)}$.

13. $\frac{x^2-(p+q)x+pq}{x^2-(p+r)x+pr}$.

14. State which of the following can be simplified by striking out each a^2:

$$\frac{a^2x}{a^2y},\quad \frac{a^2+x}{a^2+y},\quad \frac{a^2x}{a^2+y},\quad \frac{x+a^2}{5+3a^2},\quad \frac{a^2(x+y)}{5a^2}.$$

Reduce to a mixed quantity:

15. $\frac{x^2-3x-5}{x}$.

16. $\frac{8a^2-b}{2a+1}$.

17. $\frac{x^3-x}{x^2+x+1}$.

Reduce to an improper fraction:

18. $3x-2+\frac{5}{x}$.

20. $6a-\frac{(2-3a)^2}{4}$.

19. $x+2y-\frac{4y^2+1}{x+y}$.

21. $a^2-\left[a-\left(1-\frac{2}{a+1}\right)\right]$.

22. Reduce $\frac{100a+10b+c}{3}$ to a mixed number. Show that the result obtained proves that if the sum of the digits of any given number is divisible by 3, the number itself is divisible by 3.

Treat $\frac{100a+10b+c}{9}$ in like manner and state the property thus proved concerning the divisibility of a number by 9.

23. Define *fraction*. *Improper fraction*. *Mixed number*.

24. Define *integral expression*. *Fraction at its lowest terms*.

25. State some of the uses of fractions.

26. The labor of writing $\frac{9ab-12b^2}{12a^2-16ab}$ is about how many times that of writing $\frac{3b}{4a}$? What principle makes this economy possible?

Supply the missing numerator in each of the following:

27. $\frac{5}{3x-3}=\frac{?}{6x^2-6}$.

28. $\frac{3}{x-2}=\frac{?}{x^4-8x}$.

Reduce the following to equivalent fractions having their least common denominator:

29. $\frac{2}{3a^2bc}, \frac{5}{6ab^2c}, \frac{8}{abcd}$.

30. $\frac{x+2}{2x^2+x-1}, \frac{x-3}{4x^2-1}$.

31. $\frac{3a^2}{a^3-b^3}, \frac{2a}{a-b}, \frac{5a}{a^2+ab+b^2}$.

18. Addition and Subtraction of Fractions.

Ex. Simplify $\frac{a^2+b^2}{a^2-b^2}-\frac{a}{a+b}+\frac{b}{b-a}$.

$$\frac{a^2+b^2}{a^2-b^2}-\frac{a}{a+b}-\frac{b}{a-b}=\frac{a^2+b^2-a(a-b)-b(a+b)}{a^2-b^2}$$

$$=\frac{a^2+b^2-a^2+ab-ab-b^2}{a^2-b^2}$$

$$=\frac{0}{a^2-b^2}=0. \quad \textit{Ans.}$$

Let the pupil check the work by letting $a=2$ and $b=1$.

EXERCISE 31

Simplify and check:

1. $\frac{2x-3}{5}-\frac{x-1}{10}+\frac{3(2-x)}{15}$.

2. $\frac{4x-7}{7x^2}-\frac{7x-3}{9x}+\frac{3x-2}{3x^3}$.

3. $\frac{2}{a+3}-\frac{4}{a+2}+\frac{3}{a+1}$.

4. $\frac{y}{x+y}-\frac{xy}{(x+y)^2}-\frac{xy^2}{(x+y)^3}$.

5. $\frac{3a+b}{a-2b}-\frac{3a-b}{a+2b}+\frac{14ab}{4b^2-a^2}$.

6. $\frac{2}{r+4}-\frac{r-3}{r^2-4r+16}-\frac{r^2}{r^3+64}$.

7. $\frac{b}{4+4b^2}-\frac{b}{2+2b^4}+\frac{1}{8-8b}-\frac{1}{8+8b}$.

8. $\frac{x-4}{2x-3}-\frac{3x-1}{x+2}+\frac{5x^2-9x+11}{2x^2+x-6}$.

9. $\frac{1}{(5-a)(a-3)}+\frac{1}{10-7a+a^2}-\frac{1}{(a-2)(3-a)}$.

10. $\frac{p+1}{6p-6}-\frac{2p-1}{12p+12}+\frac{3}{3-3p^2}-\frac{7}{12p}$.

11. By substituting numerical values for both x and y, determine whether

$$\frac{2x}{x^2-y^2}+\frac{2}{y-x}+\frac{1}{x+y} \text{ equals } \frac{x-3y}{x^2-y^2}.$$

12. (1) In the formula $A=\frac{3l}{l^2-1}+\frac{4}{1-l}+\frac{1}{1+l}$, find the value of A when $l=10$. Also when $l=11$. When $l=12$.

(2) Now add the three fractions in the formula and, by substituting in the result, find the three required values of A.

(3) Compare the amount of work in the two processes. What does this comparison teach us as to the efficiency value of the addition of fractions?

19. Multiplication and Division of Fractions.

Ex. Simplify $\frac{a^3+b^3}{a^3+a^2b+ab^2}\times\left(1+\frac{b}{a-b}\right)\div\frac{a^2-ab+b^2}{a^3-b^3}$.

$$\text{The expression}=\frac{(a+b)(a^2-ab+b^2)}{a(a^2+ab+b^2)}\times\frac{a}{a-b}\times\frac{(a-b)(a^2+ab+b^2)}{a^2-ab+b^2}$$

$$=a+b. \quad \textit{Ans.}$$

Let the pupil check the work by letting $a=2$ and $b=1$.

EXERCISE 32

Simplify and check:

1. $\frac{x^6-y^6}{(x-y)^2}\div\frac{x^2+xy+y^2}{x-y}$.

2. $\frac{x^3+1}{8}\div\left(1-\frac{1}{x}+\frac{1}{x^2}\right)$.

3. $\left(b-a+\frac{a^2}{b}\right)\div\left(\frac{a}{b^2}+\frac{b}{a^2}\right)$.

4. $\frac{28\,ax^2}{15\,b^2c}\times\frac{9\,a^2b}{8\,c^2x}\div\frac{21\,a^3x}{10\,bc^3}$.

5. $\frac{p^2-16\,q^2}{p^2-25\,q^2}\div\frac{p^2+2\,pq-8\,q^2}{2\,p^2-9\,pq-5\,q^2}$.

6. $\frac{(x+y)^3}{2\,x(x-y)^2}\times y^2\left(x-\frac{y^2}{x}\right)\div\left(1+\frac{y}{x}\right)$.

7. $\left(a+1+\frac{1}{a}\right)\left(a-1+\frac{1}{a}\right)\div\frac{a^6-1}{a^2(a^2-1)}.$

8. $\frac{a-b-c}{a+b-c}\times\frac{b^2-(a-c)^2}{a^2-(b+c)^2}\div\frac{c^2-(a-b)^2}{(a+b)^2-c^2}.$

9. $\left(\frac{a+2\,b}{a-2\,b}+\frac{a-2\,b}{a+2\,b}\right)\div\left(\frac{a+2\,b}{a-2\,b}-\frac{a-2\,b}{a+2\,b}\right).$

10. $\left(a-1+\frac{5}{a+1}\right)\left(\frac{a^2}{6}+\frac{a}{4}+\frac{1}{12}\right)\div\left(\frac{2\,a^3+8\,a}{3\,a+3}\right).$

11. $\left[x^2-y^2+\frac{4\,xy(y+x)}{x-y}\right]\div\left[\frac{x^2+y(y+2\,x)}{4\,x^2-6\,xy+2\,y^2}\right].$

12. $\left(2+\frac{m}{m-3}\right)\left(\frac{9-m^2}{4-m^2}\right)\left(\frac{2-m}{m^2+m-6}\right)-\frac{2}{m+2}.$

13. Define the reciprocal of a number. What is the reciprocal of 2? Of $\frac{3}{5}$? Of x? Of $\frac{a}{b}$?

14. By substituting numerical values for a, b, x, and y, determine whether

$$\frac{a^2-b^2}{x+y}\cdot\frac{a^2}{(x-y)^2}\div\frac{a-b}{x^2-y^2}\quad\text{equals}\quad\frac{a^2(a+b)}{x-y}.$$

15. (1) In the formula $s=\left(t+\frac{1}{t-1}\right)\left(\frac{2\,t-2}{t^3+1}\right)$, find the value of s when $t=5$. Also when $t=22$. Also when $t=1.5$.

(2) Now reduce the formula to its simplest form. By substituting in this last result, find the three required values of s.

(3) Compare the amount of work in the two processes. What does the comparison teach us as to the efficiency value of the multiplication of fractions?

20. A Complex Fraction is one having a fraction in its numerator, or in its denominator, or in both.

In simplifying any complex fraction, it is important to write the entire fraction at each step of the process.

Ex. 1. $$\frac{1-\frac{2}{3x}}{x-\frac{4}{9x}}=\frac{\frac{3x-2}{3x}}{\frac{9x^2-4}{9x}}=\frac{3x-2}{3x}\cdot\frac{9x}{9x^2-4}=\frac{3}{3x+2}.\quad \textit{Ans.}$$

When the numerator and denominator of a complex fraction each contains one or more fractions, the expression is often simplified most readily if we

Multiply both numerator and denominator by the lowest common denominator of all the fractions contained in them.

Ex. 2. Simplify $\dfrac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}$.

Multiplying both numerator and denominator by xyz, we obtain

$$\frac{yz+xz+xy}{x^2z+xy^2+yz^2}.\quad \textit{Ans.}$$

EXERCISE 33

1. $\dfrac{\frac{4}{x}-x}{1+\frac{x}{2}}$.

2. $\dfrac{2-\frac{1}{x}}{4-\frac{1}{x^2}}$.

3. $\dfrac{x-\frac{1}{x}}{1-\frac{1}{x}}$.

4. $\dfrac{\frac{ab}{c}-2d}{a-\frac{2cd}{b}}$.

5. $\dfrac{1-\frac{1}{a+1}}{1+\frac{1}{a-1}}$.

6. $\dfrac{x-\frac{2}{3}}{6}$.

7. $\dfrac{\frac{1}{x}-\frac{3}{x^3}-\frac{2}{x^2}}{\frac{9}{x^2}-1}$.

8. $\dfrac{\frac{2x}{y}+1-\frac{y}{x}}{\frac{2x}{y}+\frac{y}{x}-3}$.

9. $\dfrac{1-\frac{2x-2z}{x+y-z}}{1+\frac{2z}{x-y-z}}$.

10. $\dfrac{2}{1-\frac{1}{a^2}}-\dfrac{3}{1-\frac{1}{a}}$.

11. $\dfrac{\dfrac{x}{1+x}+\dfrac{1-x}{x}}{\dfrac{x}{1+x}-\dfrac{1-x}{x}}.$ **12.** $\dfrac{1-\dfrac{1}{a}}{1-\dfrac{1}{a^2}}-\dfrac{2}{a^2-a}.$

Find the value of

13. $\dfrac{5a-3b}{4a-5b}$, when $a=\frac{1}{2}$ and $b=\frac{2}{3}$.

14. $\dfrac{3x+2y}{5x-y}$, when $x=\frac{2}{5}$ and $y=-\frac{2}{3}$.

15. Does $\dfrac{3x-7}{2x-5}$ equal $\frac{2}{5}$ when $x=\frac{2}{3}$?

FRACTIONAL EQUATIONS

21. An Equation is a statement of the equality of two algebraic expressions.

Define *members* of an equation and illustrate by an example.

A *root* of an equation is a number which, when substituted for an unknown quantity in that equation, satisfies the equation; that is, reduces the two members of the equation to the same number.

Thus, 6 is the root of the equation $3x-1=2x+5$, since, when 6 is substituted for x, the given equation reduces to $17=17$.

Define *degree of an equation* (of one unknown), *simple equation, identity, conditional equation, fractional equation, numerical equation, literal equation.*

22. Aids in Solving Equations. The following principles are frequently used as aids in solving equations:

The roots of an equation are not changed if

(1) *The same quantity is added to both members of the equation;*

(2) *The same quantity is subtracted from both members of the equation;*

(3) *Both members are multiplied by the same quantity or by equal quantities* (provided the multiplier is not zero, or an expression containing the unknown);

(4) *Both members are divided by the same quantity* (provided the divisor is not zero, or an expression containing the unknown).

Other principles similar are used later as aids in solving equations.

Explain the meaning of *transposition,* and show for which of the above principles it is an abbreviation.

Ex. 1. Solve $\frac{4}{3+x}-\frac{3}{x-3}=\frac{8x+3}{9-x^2}$.

Rewriting the second fraction (why?),

$$\frac{4}{3+x}+\frac{3}{3-x}=\frac{8x+3}{9-x^2}.$$

Multiplying each member by $9-x^2$, the L. C. D. of the fractions involved (see § 22, (3)),

$$4(3-x)+3(3+x)=8x+3$$

$$12-4x+9+3x=8x+3$$

$$-4x+3x-8x=-12-9+3$$

$$-9x=-18$$

$$x=2. \quad \textit{Root.}$$

CHECK. $\frac{4}{3+x}-\frac{3}{x-3}=\frac{4}{3+2}-\frac{3}{2-3}=\frac{4}{5}+3=\frac{19}{5}$.

$\frac{8x+3}{9-x^2}=\frac{16+3}{9-4}=\frac{19}{5}$.

Ex. 2. Solve $\frac{2a}{3}\left(\frac{x}{a}-a\right)=\frac{3a}{5}\left(\frac{x}{a}+a\right)$ for x.

Removing the parentheses, $\frac{2x}{3}-\frac{2a^2}{3}=\frac{3x}{5}+\frac{3a^2}{5}$.

Hence, $10x-10a^2=9x+9a^2$

$x=19a^2$. *Root.*

Let the pupil check the work.

EXERCISE 34

Solve for x and check:

1. $2x-8-\frac{24-2x}{7}=0.$

2. $\frac{2}{3}(1+x)-\frac{3}{4}(2x+1)=0.$

3. $x+2=x-8-2\{8-3(5-x)-x\}.$

4. $\frac{3x-1}{7}-\frac{x+1}{6}-\frac{1}{21}(4x+1)=\frac{3}{4}(x-1)-3.$

5. $\frac{6.3x+26.15}{8.3}=5.2.$

6. $\frac{3.1}{3x-9.5}=\frac{2.4}{2x-4.7}.$

7. $\frac{3.2x-3.4}{4.5}=\frac{.6x+4}{2.5}.$

8. $\frac{x-a}{b-a}=2-\frac{x-c}{b-c}.$

9. $(a+x)(b+x)-\frac{a^2c+bx^2}{b}=a(b+c).$

10. $\frac{2}{x+2}-\frac{1}{x-1}-\frac{1}{x+1}=0.$

11. $\frac{3}{x-2}+\frac{1-3x}{x^2+2x+4}=\frac{1-x}{8-x^3}.$

12. $\frac{x-3}{2x+6}-\frac{x^2+6}{6x^2-54}-\frac{x+3}{3x-9}=0.$

13. $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}=\frac{4ab}{4b^2-x^2}.$

14. $\frac{3-x}{1-x}-\frac{5-x}{7-x}=1-\frac{x^2-2}{7-8x+x^2}.$

15. $\frac{x^2-a^2}{x-a}+\frac{x^2-b^2}{x-b}+\frac{x^2-c^2}{x-c}=a+b+c-4x.$

16. $\frac{1}{p(q-x)}+\frac{1}{q(r-x)}-\frac{1}{p(r-x)}=0.$

17. $\frac{3x^2-5}{3x-6}-(2+x)-\frac{7}{6x+12}-\frac{7}{2x^2-8}=0.$

18. $\frac{a}{x}-\frac{b}{a}-\frac{a^2-b^2}{a^2+ab}=\frac{a+b}{x}.$

Find the value of the letter in each of the following:

19. $\frac{4a-7}{6a+18}=\frac{7a+2}{10a+30}-\frac{11}{45}.$

20. $\frac{5}{s-7}-\frac{7}{s+3}-\frac{5s+1}{s^2-4s-21}=0.$

21. Solve $R=a\left(\frac{1}{c}+d\right)$ for d.

22. Solve $R=\frac{ae}{c}(1+cd)$ for c.

23. Solve $R_1=R\left(\frac{2b_1}{b_2+b_3}-1\right)$ for b_2.

24. Evaluate $.907\left(\frac{D}{d}-.94\right)^2+3.7$ when $D=3$ and $d=\frac{1}{4}$.

25. In $H=\frac{plan}{33,000}$, when $H=1.23$, $p=40$, $l=\frac{4}{3}$, $a=63.62$, find n to the nearest tenth.

26. In $P=\frac{Er}{Ir-E}$, find r, to the nearest hundredth, when $P=40$, $E=4$, $I=1.5$.

27. (1) In $C=\frac{5}{9}(F-32)$, find F, when $C=200$. Also when $C=213$. Also when $C=-150$.

(2) Now solve the formula for F and thus obtain $F=\frac{9\,C}{5}+32$.

By substituting in this last result, find the three required values of F.

(3) Compare the amount of labor in the two processes. What inference do you make?

23. Two Equivalent Equations are equations which have identical roots; that is, each equation has all the roots of the other equation and no other roots.

Thus, $x^3-x^2-6\,x=0$ and $x(x+2)(x-3)=0$ are equivalent, since each is satisfied by the values $x=0$, 3, -2, and by no other values of x.

If we multiply the two members of an equation by the same expression, the resulting members are equal, but the resulting equation may not be equivalent to the original equation.

Thus, if we take the equation $x=5$ and multiply each member by $x-3$, we obtain $x\,(x-3)=5(x-3)$ or

$$(x-3)(x-5)=0,$$

which is not equivalent to the original equation, since it has the root $x=3$, which the original equation does not have.

In general, *if the two members of an integral equation are multiplied by* $x-a$, *the root* a *is introduced* and the resulting equation is not equivalent to the original equation.

24. An Extraneous Root is a root introduced into an equation (usually unintentionally) in the process of solving the equation.

The simplest way in which an extraneous root may be introduced is by multiplying both members of an integral equation by an expression containing the unknown number. See the example in § 23.

A more common way in which extraneous roots are introduced during a solution—and one more difficult to detect—is by a failure to reduce to its lowest terms a fraction contained in the original equation.

Thus, in solving $\frac{2x-6}{(x+1)(x-3)}=1$, the first step should be to reduce the fraction $\frac{2x-6}{(x+1)(x-3)}$ to its lowest terms. If this is done, we obtain the equation $\frac{2}{x+1}=1$, whence $x=1$.

If, however, we should fail to reduce the fraction to its lowest terms and should multiply both members by $(x+1)(x-3)$, we obtain $2x-6=x^2-2x-3$, whence $x^2-4x+3=0$. or

$$(x-1)(x-3)=0, \quad \text{and} \quad x=1, 3.$$

On testing both of these results, we find that the extraneous root 3 has been introduced.

Often the fraction which can be reduced to simpler terms occurs in a disguised and scattered form. In this case it is best to solve the equation without attempting to collect the parts of the fraction. An extraneous root may then be detected by checking the results obtained.

Thus, the fraction in the above equation might be changed in the following way so as to make it difficult to detect its presence in the equation:

We have $$\frac{2x-6}{(x+1)(x-3)}=1.$$

Whence, $$\frac{2x+2}{(x+1)(x-3)}-\frac{8}{(x+1)(x-3)}=1.$$

Whence, $$\frac{2}{x-3}-\frac{8}{(x+1)(x-3)}=1.$$

There is nothing in the appearance of this last equation to indicate that it contains a fraction which should be simplified before proceeding with the solution proper.

Hence, it is important constantly to remember that a root of an equation is its *root* because it *satisfies the original equation*, not because it is the *result of a series of operations*, as clearing an equation of fractions, transposition, etc.

25. Losing Roots in the Process of Solving an Equation.

If both members of the equation $(x+3)(x-2)=0$, are divided by $x-2$, we obtain $x+3=0$.

The resulting equation is not equivalent to the original equation since it does not contain the root $x=2$, which the original equation contains.

Hence, in general,

If both members of an equation are divided by an expression containing the unknown quantity, write the divisor expression equal to zero, and obtain the roots of the equation thus formed as part of the answer for the original equation.

EXERCISE 35

1. Multiply each member of the equation $x+2=1$ by $x-3$. Is the resulting equation equivalent to the original equation? Why?

2. Multiply each member of the equation $x=3$ by $x-2$. Is the resulting equation equivalent to the original equation? Why?

3. Divide each member of the equation $x^2-16=x-4$ by $x-4$. Is the resulting equation equivalent to the original equation? Why?

4. Make up and work an example similar to Ex. 4.

5. Solve the equation $\frac{x-2}{x^2-4}=1$ after first reducing the fraction to its lowest terms. Now solve the equation without reducing the fraction to its lowest terms. Do the two methods of solution give the same result? Which result is correct? Why?

6. $\frac{4}{x+2}=\frac{x^2}{x+2}-\frac{x}{2}.$

7. $\frac{x^3-8}{x^2-4}-3=x.$

8. $\frac{15}{(x+3)(x-2)}=1-\frac{3}{x+3}.$

9. $\frac{2}{(x-1)(x-2)}=\frac{2}{x-2}-1.$

10. Form an equation in which 4 is the extraneous root.

EXERCISE 36

ORAL

Solve orally for x:

1. $bx=\frac{1}{a}.$
2. $4=\frac{1}{x}.$
3. $ax-c=d.$
4. $\frac{a}{x}=1.$
5. $\frac{4}{x}=\frac{2}{5}.$
6. $-3x=\frac{9}{5}.$
7. $px=5-qx.$
8. $c=a+dx.$
9. $\frac{a}{bx}=c.$
10. $a+b=\frac{c+d}{x}.$
11. $\frac{p-q}{4}=\frac{a+b}{x}.$
12. $\frac{3}{5}x=\frac{5}{9}.$

13. If n represents a number, what does $n(n+1)(n+2)$ represent?

14. Begin with $x+3$ and count backward five numbers.

15. A rectangular field is $6x$ feet long and has a perimeter of $20x$ feet. What is the width of the field? The area?

16. If a side of a given square field is s feet, what is the increase in the area of the field when its side is increased by 10 ft.?

17. When sugar is at a cents a pound, how many more pounds can be bought for a dollar than when it is b cents a pound, b being greater than a?

18. If a barrels of flour are sold for b dollars at a gain of c dollars a barrel, what is the cost per barrel?

19. After one railroad train had traveled $1\frac{1}{2}$ hours, another train followed it and overtook it in x hours. How many hours did the first train go? If the first train went at the rate of 20 miles an hour, how many miles did it travel?

20. Express the following as an equation: x divided by 16 gives 9 for a quotient and 5 for a remainder.

21. A pupil has worked 20 problems and x of them are right. What is his average? If he should work 10 more problems and 8 of them are right, what would be his new average?

22. A baseball nine has played 42 games and won 30 of them. If it should play x more games and win $\frac{3}{4}$ of them, what would be its new average?

23. If milk is 6 % butter fat, how many gallons of butter fat are in 50 gallons of milk? If x gallons of water are added to the milk, what per cent of the resulting mixture will be butter fat?

24. A boy's average in four of his studies is .885. If his grade in a fifth study is x, what is his average in all five studies?

25. How many hours will it take a man to walk x miles out into the country at the rate of three miles an hour, and to ride back at 8 miles an hour?

26. Express 4 % of $\frac{3}{5}$ of x added to $3\frac{1}{2}$ % of $\frac{2}{5}$ of x.

EXERCISE 37

1. New York and Philadelphia are 90 miles apart. Two bicyclists start from these places at the same time, travel toward each other, and meet in $7\frac{1}{2}$ hours. If one of them travels 2 miles an hour faster than the other, what is the rate of each?

2. A, B, and C together have \$1285. A's share is \$25 more than $\frac{5}{6}$ of B's, and C's share is $\frac{4}{15}$ of B's. Find the share of each.

3. A pupil has worked 20 problems. If he works 10 more and gets 8 of them right, his average will be .80. How many problems has he worked correctly thus far?

4. A ball nine has played 42 games and won 30. If after this it should win $\frac{3}{4}$ of the games played, how many games must it play to bring its average up to .72?

5. How much water must be added to 50 gallons of milk containing 6 % of butter fat to make a mixture containing 4 % of butter fat?

6. A mass of silver and copper alloy weighs 128 lb. and contains 12 lb. of silver. How many pounds of silver must be added to the mass in order that 8 lb. of the resulting alloy shall contain 1 lb. of silver?

7. A boy's average in four of his studies is .885. What grade must he get in a fifth study to make his average .90?

8. If the wheat crop of the United States during a period of 5 years averaged 680 millions of bushels, what must it average during the next two years to bring the average of the 7 years up to 700 millions of bushels?

9. If 100 lb. of sea water contains $2\frac{1}{2}$ lb. of salt, how much water must be evaporated from 200 lb. of sea water in order that 12 lb. of the water remaining shall contain 1 lb. of salt?

10. How much water must be added to 2½ gal. of alcohol which is 90 % pure to make a mixture 80 % pure?

11. If a certain man can spade a garden in 2 days and a boy can spade it in 6 days, how long will it take the man and the boy together to spade the garden?

12. If A, B, and C together can do in 10⅔ days a piece of work which B alone can do in 48 days, or C alone in 32 days, how long will it take A alone to do the work?

13. One pipe can fill a swimming pool in 24 min. and another pipe can fill the same tank in 36 min. How long will it take the pipes together to fill the tank?

14. Make up and work a similar example concerning two pipes which fill a given tank and another pipe which at the same time empties the tank.

15. A man having 10 hr. at his disposal rides out into the country at the rate of 10 mi. an hour and walks back at the rate of 3 mi. an hour. Find the distance he rides.

16. A man invests ⅗ of his capital at 4 % and the rest at 3½ %. The income from these investments is $304. Find his capital.

17. A freight train started from a certain station and proceeded at the rate of 20 miles an hour. After 1½ hours it was followed by a passenger train going 40 miles an hour. In how many hours will the second train overtake the first?

18. A certain number when divided by 16 gives 9 for a quotient and 5 for a remainder. Find the number.

19. Four men together made $19,200 in one year. The first made ⅔ as much as the fourth, the second ⅚ as much as the fourth, and the third 50 % more than the fourth. How much did each make?

20. A farmer has 280 yd. of wire fence and wishes to make a cattle yard which shall be 2½ times as long as it is wide. Find the dimensions of the yard.

21. If 30 lb. of water at 150° are mixed with 20 lb. of water at 50°, what will be the temperature of the resulting mixture?

Sug. Take as a unit of heat the amount of heat required to increase the temperature of 1 lb. of water by 1°. Then,

NUMBERS DEALT WITH	SYMBOLS
(1) Temperature of the mixture	$=x$.
(2) Number heat units lost by the 30 lb. water	$=30\ (150-x)$.
(3) Number heat units gained by the 20 lb. water	$=20\ (x-50)$.

Then (2) must equal (3), or

$$30\ (150-x)=20\ (x-50), \text{ etc.}$$

Hence, $50\,x=4500+1000$, etc.

22. If 100 lb. of water at 212° are mixed with 40 lb. at 32°, find the temperature of the mixture.

23. How many quarts of water at 120° must be mixed with 80 quarts at 50° to produce a mixture having a temperature of 100°?

24. A crew can row 6 miles an hour in still water. If it takes the crew 4 hours to row 12 miles up-stream, find the rate of the current.

25. In a bookcase the distance between the bottom and top boards is 56 in. It is desired to insert in this space 5 shelves each 1 in. thick, so that the book spaces thus formed shall increase by 1 in. successively from top to bottom. Find the distances between the successive shelves.

EXERCISE 38

Oral

1. If the product of two numbers is p, and one of the numbers is a, what does p/a represent?

2. If c represents a certain number of cents what does $c/100$ represent? $c/25$? $c/10$?

3. A man has traveled x miles at y miles an hour. What does x/y represent?

4. Find the length of a piece of carpet 27 in. wide that will contain y square yards?

5. If a house rents for d dollars per month, what does $12d$ stand for?

6. If a man can do a piece of work in x days, what is represented by $1/x$? By $5/x$?

7. If the dimensions of a rectangle are x and y feet, what is represented by $2x+2y$? By $\sqrt{x^2-y^2}$? By xy?

8. If a side of a square is x, what does $(x+4)(x-3)$ stand for?

9. Three boys can run the quarter mile in a, b, c seconds respectively. What does $(a+b+c)\div 3$ represent?

10. Each side of a square contained f feet. From the square was cut out a smaller square each side of which was i inches. Express in square feet the area left.

11. How many board feet are in a stick of lumber a feet long, b inches wide, and c inches thick?

12. If a number n be diminished by r per cent of itself, how much of the number is left?

13. What is the cost of manufacturing n articles at d dollars per hundred articles?

14. If the dimensions of a school room are l, w, h feet and 200 cu. ft. of space are allowed for each pupil, what is represented by $lwh \div 200$?

EXERCISE 39

1. If an agent sells a property for s dollars and receives a commission of r per cent; obtain a formula for b, the amount which the agent remits to the owner of the property.

Solve this formula for s. By aid of this last result, determine the selling price, when the commission is 5 %, in order that $1900 be remitted.

2. In Fig. 1 and 2, all the angles are right angles. Obtain a formula for the area of Fig. 1. Also for Fig. 2.

3. Fig. 3 represents a square plate (with side *a*), in which four square notches have been cut, each side of which is *b*. Find a formula for the area left.

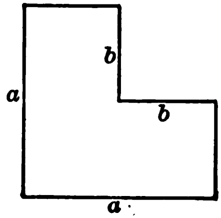


FIG. 1.

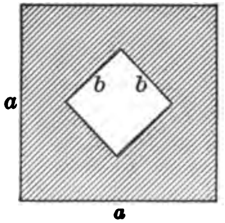


FIG. 2.

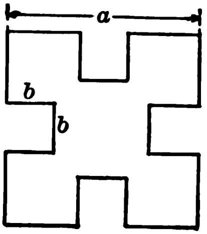


FIG. 3.

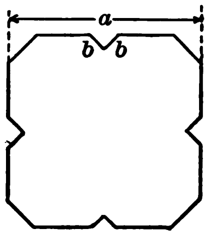


FIG. 4.

4. Fig. 4 is a figure that was originally a square with a side *a*, but from which notches have been cut, and from which the four corners have been cut off, all the eight parts removed being equal. Find a formula for the area of the figure as it stands.

5. In the shortest way (by the aid of factoring) by use of the formula obtained in Ex. 3, find the area of Fig. 4, when *a* is 3.36 in. and *b* is .32 in.

6. Obtain a formula for the cost (C) in dollars of *n* sticks of lumber, each *l* feet long, *w* inches wide, and *t* inches thick, at *d* dollars per thousand feet.

Make up and work a numerical application of this formula.

7. If *p* denotes the principal, *r* the rate of interest expressed decimally, and *n* the number of days, state the formula for *i*, the common interest (i.e., regarding 360

days as a year) and also for i', the exact interest (i.e., regarding 365 days as a year).

Hence, find the formula for $i-i'$ and reduce it to its simplest form.

8. By use of the formula obtained in Ex. 7, find the difference between the exact interest and the common interest on $2520 for 67 days at 6 %.

9. In manufacturing a certain article, the cost of the preparatory work was m dollars, after which the cost of manufacturing articles was b cents each. Obtain a formula for the entire cost of manufacturing n objects. Also a formula for the average cost in cents (C) of manufacturing each of the n objects.

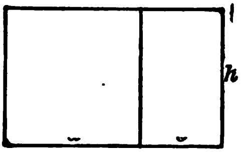
Illustrate the use of this formula by a numerical example.

10. In a given city the population at the beginning of the year was p, the birth rate and death rate during the year were b and d per 1000 respectively, the accessions to the population from outside were a persons, and the removals were b persons. Obtain a formula for P, the population in the city at the end of the year.

Illustrate this formula by a numerical example.

11. Using appropriate letters, obtain a formula for converting a given number of hundred weight of grain into bushels, weighing 60 pounds each.

Make up a numerical example and solve it by use of this formula.



12. Show that $ah+bh$ may be represented by the adjoining diagram.

13. Draw a similar figure to represent $ah+bh+ch$.

14. Also to represent $2\,ah+ch$.

15. Also $2\,ah+2\,ch$. **16.** Also $2\,ah+2\,ch+ac$.

EXERCISE 40

REVIEW

1. At sight write the product of

(1) $(ax^{n-1}-x^{3}y^{2m})^2$.

(2) $(.3\,x-1.4\,y)^2$.

(3) $(3\,a-4\,b)(5\,a+.3\,b)$.

(4) $(a^2+ax+x^2)(a^2-ax+x^2)$.

2. Find the value of $\frac{9\,pq-12\,q^2}{12\,p^2-16\,pq}$, when $p=5$ and $q=2$, both before and after reducing the fraction to its simplest form. Compare the amount of work in the two evaluations.

3. The perimeter of a rectangle is 208 ft. The width is $\frac{5}{8}$ of the length. Make a drawing of the rectangle to scale.

4. Simplify $3(.3x-.2y)^2-2(.3x+.2y)^2$.

5. A tractor plows an acre in m minutes. Find a formula for the number of hours (H) which it will require to plow a field l yards long and w yards wide.

6. For the eight half years of their high school course, three pupils, James Ashley, Frank Evans, and William Porter, made general averages as follows:

Ashley	67	71	74	81	83	85	87	88
Evans	82	88	86	81	78	84	86	91
Porter	84	73	77	69	78	83	74	77

Graph the three sets of grades on the same chart.

Which of the three pupils made the greatest improvement? Which of them made the most uniform improvement?

7. State which of the following processes increase the value of a fraction: (1) multiplying the numerator by 3; (2) dividing the denominator by 3; (3) multiplying the denominator by 3. Illustrate numerically.

8. If it takes f cubic feet of coal to make a ton, obtain a formula for T, the number of tons a bin $a\times b\times c$ feet will hold.

9. Express the sum of $\frac{5}{x-2}$ and $\frac{6}{x-3}$ as a single fraction.

10. The following tabulation gives the population of the United States in millions, and wheat crops in millions of bushels:

YEAR	1870	1880	1890	1900	1910	1920
Population...........	39	50	63	76	92	
Wheat Crop..........	236	499	399	522	635	
Per capita............						

Find the per capita production and graph the three sets of figures. Can you draw any important inference from your chart?

11. Factor: (1) $x^{6n}-y^{4n}$. (4) $40\,x^3-5$.
(2) $.09-a^2-(b^2-2\,ab)$. (5) x^3-x+y^3-y.
(3) $x^3-3\,x+2$. (6) $14\,x^2-3\,xy-5\,y^2$.

12. Solve and check: $\dfrac{6\,x+6}{2\,x^2+5\,x+3}+\dfrac{2\,x+1}{1+x-2\,x^2}-\dfrac{2\,x}{x^2+2\,x}=0.$

13. Simplify and check: $\dfrac{b}{d}+\dfrac{ad-bc}{cd+d^2x}.$

14. Find a formula for the number of hours (H) a child n years of age should sleep each day, if the rule is as follows: to 8 hours add $\frac{1}{2}$ hour for each year that the child is under 18 years of age.

By use of this formula determine the number of hours sleep per day required by a child 12 years old. By one 15 years old. By a child 6 years old.

15. If 5 be subtracted from three fourths of a certain number, the remainder will equal the number divided by 5. Find the number.

16. A dealer sells 40 % less than the catalogue price and yet makes a profit of 25 %. What is the cost to him of an article whose catalogue price is m dollars?

17. Simplify $\left(1-\dfrac{x-y}{x+y}\right)\left(2+\dfrac{2\,y}{x-y}\right)$ and check by letting $x=2$, $y=1$.

18. Compute the reciprocal of 13 to the nearest thousandth.

19. If a pounds water at a temperature of 200° are added to b pounds at a temperature of 120°, what is the temperature of the resulting mixture?

20. What is the advantage of studying factoring before studying fractions?

21. Simplify $\frac{3a-2b}{7a-7b} \div 3\frac{1}{7}$.

22. The following tabulation gives in millions of dollars foreign (international) trade of the whole world and also of the United States in the years specified.

YEAR	1810	1820	1830	1840	1850	1860
World Trade..........	1600	1705	2035	2665	4160	7445
U. S. Foreign Trade...	152	143	134	222	318	687
Per cent..............						

YEAR	1870	1880	1890	1900	1910	1920
World Trade..........	10,955	15,165	16,800	21,500	32,400	
U. S. Foreign Trade...	829	1,504	1,680	2,244	3,429	
Per cent..............						

At each date given, determine what per cent the foreign trade of the United States is of that of the whole world and complete the tabulation. Then graph on one chart the three sets of figures.

From the chart determine whether the foreign trade of the United States is increasing more or less rapidly than the world trade.

23. Solve and check: $\frac{x^2}{x^3-64} - \frac{1}{x-4} + \frac{2}{x^2+4x+16} = 0$.

24. If the regular number of hours in a day's work be denoted by r, and the number of hours worked overtime by v; and if the regular rate of wages per hour is c cents, and a fifty per cent higher rate is paid for overtime, obtain a formula for D, the number of dollars in a day's wages. Illustrate the use of your formula by a numerical example.

25. Solve $\frac{a-b}{x-c} - \frac{a+b}{x+2c} = 0$, for x and check.

26. How much water must be added to a pint of ammonia which is 25 per cent pure to make a mixture 10 per cent pure?

CHAPTER IV

SIMULTANEOUS EQUATIONS; GRAPHS

26. Simultaneous Equations are a set or system of equations in which more than one unknown quantity is used, and the same symbol stands for the same unknown number.

Thus, in the group of three simultaneous equations,

$$\begin{aligned} x+\ \ y+2z&=13,\\ x+2y+\ \ z&=0,\\ 2x+\ \ y-\ \ z&=3. \end{aligned}$$

x stands for the same unknown number in all of the three equations, y for another unknown number, and z for still another.

27. Independent Equations are those which cannot be derived one from the other.

Thus, $x+y=10$,

and $2x=20-2y$,

are not independent equations, since by transposing $2y$ in the second equation and dividing it by 2, we may convert the second equation into the first.

But $\left.\begin{aligned} 3x-2y&=5\\ 5x+\ \ y&=6 \end{aligned}\right\}$ are independent equations, since neither one of them can be converted into the other.

28. Elimination is the process of combining two equations containing two unknown quantities so as to form a single equation with only one unknown quantity. Or, in general, elimination is the process of combining several simultaneous equations so as to form equations one less in number and containing one less unknown quantity.

29. Methods of Elimination.

Ex. 1. By the method of *addition and subtraction*, solve

$$9x-8y=5. \quad \ldots\ldots \quad (1)$$

$$15x+12y=2. \quad \ldots\ldots \quad (2)$$

The L. C. M. of 8 and 12 is 24.

Multiply in (1) by $24\div 8$, or 3; and in (2) by $24\div 12$, or 2.

Hence,
$$\begin{array}{r} 27x-24y=15 \\ \underline{30x+24y=\ \ 4} \end{array}$$

By adding, $57x=19$, or $x=\frac{1}{3}$. *Root.*

Substituting $\frac{1}{3}$ for x in (1), $3-8y=5$, or $y=-\frac{1}{4}$. *Root.*

Let the pupil check the work.

Ex. 2. By the method of *substitution*, solve

$$9x-8y=5. \quad \ldots\ldots \quad (1)$$

$$15x+12y=2. \quad \ldots\ldots \quad (2)$$

From (1), $9x=5+8y$, or $x=\frac{5+8y}{9}$.

Substituting for x in (2),

$$\frac{15(5+8y)}{9}+12y=2, \quad \text{whence} \quad y=-\tfrac{1}{4}. \quad \textit{Root.}$$

Substituting for y in (1), we find $x=\frac{1}{3}$. *Root.*

Ex. 3. Solve the following for x and y:

$$(a+b)x-(a-b)y=3ab. \quad \ldots\ldots \quad (1)$$

$$(a-b)x-(a+b)y=ab. \quad \ldots\ldots \quad (2)$$

Multiplying (1) by $a+b$ and (2) by $a-b$,

$$(a^2+2ab+b^2)x-(a^2-b^2)y=3a^2b+3ab^2. \quad \ldots \quad (3)$$

$$(a^2-2ab+b^2)x-(a^2-b^2)y=a^2b-ab^2. \quad \ldots \quad (4)$$

(3) – (4), $4abx=2a^2b+4ab^2$.

Whence, $x = \frac{1}{2}(a+2b)$,
In like manner find $y = \frac{1}{2}(a-2b)$.

$$\left.\begin{aligned} x &= \tfrac{1}{2}(a+2b), \\ y &= \tfrac{1}{2}(a-2b). \end{aligned}\right\} \textit{Ans.}$$

Let the pupil check the work.

EXERCISE 41

In solving the following pairs of simultaneous equations, use that one of the two methods of elimination which is best adapted to each particular problem:

1. $3x+4y=1$.
$6x-2y=-3$.

2. $p=2q-3$.
$p=4q-7$.

3. $3x+2y=21$.
$y=2x$.

4. $x=5y-2$.
$y=2x-3$.

5. $x=6y-3$.
$8-5y=x$.

6. $y=\frac{1}{2}(x-5)$.
$y=\frac{3}{4}(2x+1)$.

7. $4m-3y=-1$.
$5m-4y=-1$.

8. $2x-3y+17=0$.
$y=3$.

9. $\dfrac{x}{4}+\dfrac{2y}{3}=1$.
$\dfrac{5x}{8}+\dfrac{3y}{2}=2$.

10. $5x-y=-1.8$.
$3x+y=11.4$.

11. $\dfrac{y}{2}-\dfrac{x}{3}=1$.
$\dfrac{y}{4}-\dfrac{x}{9}=1$.

12. $y=2x-3.4$.
$y=2.3x-6.4$.

13. $\frac{1}{3}x=2-\frac{1}{4}y$.
$3x+4y-25=0$.

14. $\frac{1}{4}x-\frac{1}{3}(y+1)=1$.
$\frac{1}{3}x-\frac{1}{4}(3y+1)=0$.

15. $\frac{1}{5}(2x-y)-\frac{1}{2}(3x-2y)=5y-2$.
$\frac{5}{6}(x-4y)=17-\frac{3}{4}(7x-2y)$.

16. $\dfrac{2x+y}{2}-\dfrac{3x+1}{7}-\dfrac{x+2y}{8}=0$.
$\frac{1}{3}(4x-2)-\frac{1}{5}(x-y)+\frac{1}{2}(5x+4y)=0$.

17. $\dfrac{7}{x+3}-\dfrac{3}{y+4}=0$.
$x(y-2)-y(x-5)=-13$.

18. $\dfrac{x}{a+b}+\dfrac{y}{a-b}=\dfrac{2}{a^2-b^2}$.
$\dfrac{x}{a-b}+\dfrac{y}{a+b}=\dfrac{2}{a^2-b^2}$.

19. $(x-1)(y+2)-(x-5)(y+3)=0$.
$xy+2x-x(y+10)=72y$.

Solve for x and y:

20. $ax+by=a^2+b^2.$
$bx+ay=2ab.$

21. $ax+by=c.$
$a'x+b'y=c'.$

22. $x+y=a+b.$
$\dfrac{a+x}{b+y}=\dfrac{b}{a}.$

23. $(c-d)x+dy=c^2.$
$dx-cy=d^2.$

24. $ax-by=\frac{1}{2}(a^2+b^2).$
$(a-b)x-(a+b)y=0.$

25. $\dfrac{a}{b+y}-\dfrac{b}{a-x}=0.$
$\dfrac{a}{b+x}-\dfrac{b}{a-y}=0.$

26. $(p+q)x+(p-q)y=p^2-q^2.$
$(p-q)x+(p+q)y=p^2+q.^2$

30. Three or More Simultaneous Equations.

Ex. Solve $4x-2y+z=-2.$ . . . . . . (1)

$3x-y-2z=11.$ . . . . . . (2)

$6x-y+3z=-3.$ . . . . . . (3)

If we choose to eliminate z first, multiply (1) by 2, and use (2) unchanged.

Then $8x-4y+2z=-4.$ . . . . (4)

$3x-y-2z=11.$ . . . . (5)

Add the corresponding members of (4) and (5).

$11x-5y=7.$ . . . . (6)

Also multiply (1) by 3, and set down (3) unchanged.

$12x-6y+3z=-6.$ . . . . (7)

Subtracting, $6x-y+3z=-3.$ . . . . (8)

$6x-5y \quad =-3.$ . . . . (9)

From (6) and (9), $5x=10$, hence $x=2$. *Root.*
By substitutions in (9) and (1), find $y=3$, and $z=-4$. *Roots.*
Let the pupil check the work.

31. Use of $\frac{1}{x}$ and $\frac{1}{y}$ as Unknown Quantities.

Ex. Solve

$$\frac{5}{8x}-\frac{3}{2y}=\frac{41}{4}. \quad \dots \dots \quad (1)$$

$$\frac{3}{4x}-\frac{5}{3y}=\frac{23}{2}. \quad \dots \dots \quad (2)$$

Multiply in (1) by 8 (the L. C. M. of 8 and 2), and in (2) by 12.

$$\frac{5}{x}-\frac{12}{y}=82. \quad \dots \dots \quad (3)$$

$$\frac{9}{x}-\frac{20}{y}=138. \quad \dots \dots \quad (4)$$

To eliminate y, multiply in (3) by 5, and in (4) by 3.

$$\frac{25}{x}-\frac{60}{y}=410. \quad \dots \dots \quad (5)$$

$$\frac{27}{x}-\frac{60}{y}=414. \quad \dots \dots \quad (6)$$

Subtracting each member of (5) from the corresponding member in (6),

$$\frac{2}{x}=4, \text{ whence } 2=4x, \text{ and } x=\tfrac{1}{2}. \text{ *Root.*}$$

Substituting for x in (3),

$$10-\frac{12}{y}=82, \text{ whence } y=-\tfrac{1}{6}. \text{ *Root.*}$$

Let the pupil check the work.

EXERCISE 42

Solve and check:

1. $\frac{2}{x}+\frac{3}{y}=1.$

   $\frac{3}{x}+\frac{4}{y}=2.$

2. $4x-5y+3z=18.$

   $5x-3y-4z=2.$

   $3x+4y-5z=32.$

3. $\frac{1}{3x}-\frac{1}{5y}=3.$
$\frac{1}{5x}+\frac{1}{2y}=8.$

4. $\frac{2}{3x}+\frac{1}{4y}=1.$
$\frac{3}{2x}+\frac{5}{8y}=2.$

5. $4x+5z=8.$
$6x-3y+2z=0.$
$2x+6y-z=-1.$

6. $4y+3x=xy.$
$2y-6x=3xy.$

7. $.8x+.4y-.5z=.5.$
$.5x-.6y+.2z=.5.$
$.9x+.5y-.6z=.5.$

8. $\frac{3a}{x}-\frac{2b}{y}=4.$
$\frac{5a}{x}-\frac{4b}{y}=7.$

9. $x-2y-2z=-2a.$
$x+y-3z=-4b.$
$3x-3y-5z=-2(a+b).$

10. $2x+y+z=2(a+b).$
$2y+x+z=2(a+c).$
$2z+x+y=2(b+c).$

11. $\frac{.4}{x}-\frac{.2}{y}=.2$
$\frac{.5}{x}+\frac{.03}{y}=1.09.$

12. $\frac{7}{.2x}-\frac{5}{.3y}=\frac{16}{3}.$
$\frac{3}{.4x}-\frac{5}{1.2y}=\frac{13}{12}.$

13. $\frac{2}{x}+\frac{3}{y}=7.$
$\frac{3}{y}+\frac{4}{z}=9.$
$\frac{5}{x}+\frac{6}{z}=15.$

14. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=6.$
$\frac{3}{x}+\frac{2}{y}+\frac{1}{z}=10.$
$\frac{3}{x}+\frac{1}{y}+\frac{3}{z}=14.$

15. $\frac{b}{x}+\frac{a}{y}=\frac{a-b}{a}.$
$\frac{a}{x}+\frac{b}{y}=\frac{b-a}{a}.$

16. $\frac{2}{3x}+\frac{3}{4y}-\frac{4}{5z}=18.$
$\frac{5}{6x}-\frac{5}{8y}+\frac{3}{4z}=-5.$
$\frac{3}{2x}-\frac{7}{5y}+\frac{3}{10z}=-41.$

17. $\frac{a}{x}+\frac{1}{y}=a^2+b.$
$\frac{1}{x}+\frac{b}{y}=a+b^2.$

32. Elimination of Letters from Formulas.

Ex. Given the formulas $V=lwh$ and $l=2w$, obtain from them a formula containing the other letters but not l.

Substituting in the first formula the value of l given in the second formula, we obtain

$$V=(2w)wh,$$

or

$$V=2w^2h. \quad \textit{Ans.}$$

EXERCISE 43

Eliminate:

1. x between $p=bx$ and $x=3d$.
2. r between $p=br$ and $r=3d$.
3. x between $v=hlx$ and $x=3h$.
4. p between $a=bp$ and $c=dp$.
5. h between $V=lwh$ and $3l=2h$.
6. s between $s=(1+g)c$ and $s=(1-d)c$.
7. c between $s=2ab+2ac+2bc$ and $c=\frac{2}{3}a$.
8. w between $V=lwh$ and $3h+5w=7$.
9. l between $a=2h(l+w)+lw$ and $5l=2w$.
10. c between $p=\frac{bc}{a}$ and $c=ad$.
11. c between $3x+2y=5c$ and $8x-2c=5$.
12. p between $3x+2y=12-5p$ and $4x-3y-2p=6$.
13. (1) By use of the two formulas $s=\frac{11c}{10}$ and $s=\frac{4a}{5}$, find a when $c=4$. Also when $c=25$. Also when $c=17$.

(2) Now eliminate s between the two given formulas and thus obtain $a=\frac{11c}{8}$. By use of this last formula, find the three required values of a.

(3) Compare the amount of work in the two methods of finding the three values of a. What inference do we draw as to the labor saving value of elimination?

EXERCISE 44

ORAL

1. If a bushel of oats was sold for a cents at a profit of 20 %, what was the original cost of the bushel of oats?

2. If x boys each contribute y dollars toward the cost of a motor boat, how much do they all contribute together? If the number of boys is increased by 2, and the number of dollars contributed by each diminished by 10, how much will be contributed?

3. How many pounds of butter fat in all are in x pounds of milk containing 4 % of butter fat and y pounds of cream containing 60 % of butter fat?

4. If gold loses $\frac{1}{19}$ and silver $\frac{1}{10}$ of its weight when immersed in water, how many pounds altogether are lost by x pounds of gold and y pounds of silver?

5. Express a number of three digits, in which the digits in order from left to right are x, y, and z.

6. Express a number of two digits, in which the tens' digit is x and the units' digit is twice as great as the tens' digit.

7. If a steamer runs 20 miles per hour in still water, how many hours will it take to run 72 miles down a stream flowing at the rate of x miles an hour?

8. If a clerk's salary of x dollars a month was increased by 25 %, how many dollars would he earn in y months at the increased salary?

9. A boy has d dollars and q quarters. What then does $100d + 25q$ represent?

10. If a school has x pupils of which b are boys, what does $x - b$ represent?

11. If a man earns d dollars and spends s dollars per month, what is the meaning of $12(d - s)$?

12. If a field is a feet long and b feet wide, what is the meaning of $ab \div 43{,}560$.

13. A woman bought a pounds of sugar at b cents a pound and c pounds of coffee at d cents a pound. She gave the clerk a five-dollar bill. What does $500-ab-cd$ mean?

14. Of 8 pupils in a class, 3 had a grade of g and 5 of h. What does $\frac{3\,g+5\,h}{8}$ stand for?

EXERCISE 45

1. The difference of two numbers is 12 and three times the smaller number exceeds twice the larger number by 18. Find the numbers.

2. A mass of metal composed of iron, lead, and aluminum weighs 5280 lb. and contains 13 cu. ft. A cubic foot of each of these three metals weighs 480 lb., 700 lb., and 156 lb., respectively, and the given mass contains 2 cu. ft. more of aluminum than of lead. How many cubic feet of each of the metals does the mass contain?

3. Five times the smaller of two numbers exceeds three times the larger by 15. Also $\frac{1}{3}$ of the larger exceeds $\frac{1}{4}$ of the smaller by 2. Find the numbers.

4. If the cost of a telegram of 13 words between two cities is 31 cents, and that of a telegram of 19 words is 43 cents, what is the charge for the first ten words in a message and for each word after that?

5. Find a fraction, such that if 1 is added to the numerator, the value of the fraction will become $\frac{2}{3}$; but if 1 is added to the denominator, the value of the fraction will be $\frac{1}{2}$.

6. Separate 3.32 into two parts such that 50 % of one part added to 18 % of the other equals .86.

7. The difference of two numbers is 28. If the larger number is divided by the smaller, the quotient is 3 and the remainder 4. Find the numbers.

8. A ton of fertilizer which contains 40 lb. of nitrogen, 120 lb. of potash, and 80 lb. of phosphate is worth $15.60. A ton containing 60 lb. of nitrogen, 110 lb. of potash, and 120 lb. of phosphate is worth $19.90, while a ton containing 55, 105, 100 lb. of each, taken in order, is worth $18. Find the value of one pound of each constituent.

9. A man has a tank containing water at a temperature of 60° and another containing water at 180°. How many gallons must be taken from each tank to make a mixture of 360 gallons at 110°?

10. If two grades of coffee, worth 18 cents and 28 cents are to be mixed to make 100 lb. which can be sold for 30 cents at a profit of 20 %, how many pounds of each must be used?

11. A farmer wishes to combine milk containing 4 % of butter fat with cream containing 60 % of butter fat in order to produce 30 gallons of cream which shall contain 30 % of butter fat. How many gallons of milk and how many gallons of cream must he use?

12. A man has a tank containing water at a temperature of t_1 degrees and also a tank containing water at t_2 degrees. How many gallons must be taken from each of these in order to make a mixture of p gallons at t degrees?

Make up and work a numerical example illustrating the use of the formula which you have obtained.

13. How much milk containing 3.4 % of butter fat must be mixed with how much cream containing 24.5 % of butter fat to make a mixture of 20 gallons containing 18 % of butter fat?

14. Of three pipes, the first and second running together can fill a swimming pool in 80 min.; the first and third can fill it in 1 hr. 40 min.; and the second and third in 2 hr. How long will it take each pipe running alone to fill it?

15. A and B together can complete a piece of work if A works 4 days and B 3 days, or if A works 2 and B 6 days.

How many days would it take each of them working alone to do the work?

16. A man has $40,000 to invest and from it wishes to obtain an annual income of $1800. If he invests part of his principal at 4 % and the rest at 5 %, how much must he invest at each of these rates?

17. A party of boys purchased a motor boat. They found that if there had been 2 more boys, they would have paid $10 apiece less; but if there had been 2 less, they would have paid $20 apiece more. How many boys were there and what did the boat cost?

18. A 16-lb. mass of gold and silver alloy, when immersed in water, weighed only 14.656 lb. If gold loses .051 of its weight and silver .095, how many pounds of each were in the alloy?

19. A mass of copper and tin weighing 300 lb. when immersed in water weighed 262.5 lb. If the specific gravity of copper is 8.8 and that of tin is 7.3, how much of each metal was there in the mass?

20. The denominator of a certain fraction exceeds the numerator by 2. If a certain number is added to both numerator and denominator, the value of the fraction thus formed is $\frac{2}{3}$; while if this number is subtracted from both numerator and denominator, the value of the fraction is $\frac{1}{2}$. Find the original fraction.

21. In a given number of three digits, the sum of the digits is 9. The units' digits exceeds the hundreds' digit by 1. Also the given number is fifty-four times as great as the sum of the units' and tens' digits. Find the number.

22. A steamer can run 20 miles an hour in still water. If the steamer can go 72 miles with a current in the same time that it can go 48 miles against the current, what is the rate of the current?

23. A clerk earned $504 in a certain number of months.

His salary was increased 25 % and he then earned $450 in two months less time than that in which he previously earned $504. What was his original salary per month?

24. One bottle contains ammonia 90 % pure and another contains ammonia 60 % pure. How much must be taken from each bottle to make a quart 80 % pure?

25. A farmer has enough feed for his oxen to last a certain number of days. If he sold 10 oxen, his feed would last 30 days longer. If, on the other hand, he were to buy 10 oxen more, his feed would last 10 days less. Find how many oxen he has and for how many days he has feed.

26. There are two alloys of silver and copper, one of which contains twice as much copper as silver, the other three times as much silver as copper; how much of each alloy is required to make an alloy of equal parts of silver and copper which will weigh $2\frac{1}{2}$ lb. avoirdupois?

27. The sum of two numbers is a, and five times the smaller number exceeds four times the larger by b. Find the numbers.

EXERCISE 46

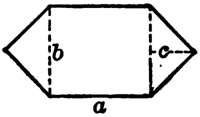


Fig. 1.

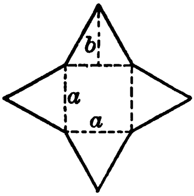


Fig. 2.

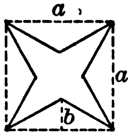


Fig. 3.

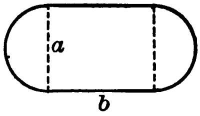


Fig. 4.

1. In Fig. 1, two isosceles triangles are attached to a rectangle. Obtain a formula for the area of the entire figure.

2. In Fig. 2, four isosceles triangles are attached to a square. Obtain a formula for the area of the entire figure.

Find a formula also for the area of the figure when the four triangles are equilateral.

3. In Fig. 3, four equal isosceles triangles have been taken from a square. Find a formula for the area of the figure left.

4. In Fig. 4, two semicircles are attached to a rectangle. Find a formula for the perimeter of the entire figure.

If $a=\frac{1}{2}b$, eliminate a from the formula. Also if $a=b$, eliminate a. Describe the shape of the resulting figure.

5. A one-half mile running-track is to be constructed in the general shape of Fig. 4, with $a=\frac{1}{2}b$. Find the length of a and b to the nearest tenth of a foot. (Use $\pi=\frac{22}{7}$.)

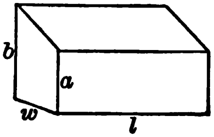
6. If a body is moving at the rate of v feet per second, obtain a formula for d, the number of miles it will go in 1 hour. State this formula as a rule. Make up a numerical example and solve it by use of this formula.

7. Solve for v the formula obtained in Ex. 6. State this new formula as a rule.

8. Construct a graph of the formula obtained in Ex. 6. By use of this graph, determine the rate in miles per hour of a boy who is traveling at the rate of 6 ft. per second. Also 20 ft. per second. Also 420 ft. in 1 minute.

9. By use of the graph constructed in Ex. 8, determine the rate in feet per second of a car that is going 20 miles per hour. Also of a boy who runs 12 miles in an hour.

10. Adjoining is the diagram of a chicken house whose end is a trapezoid. If a space of 50 cu. ft. is allowed for each hen, find a formula for N, the number of hens that can be accommodated in the chicken house.



If $a=5$, $b=8$, $w=10$, how long must the house be in order to accommodate 40 hens? (Solve by use of the formula.)

11. If s denotes the selling price of an article, c the cost, p the per cent of profit (expressed decimally), and the per cent of profit is computed on the selling price (not on the cost), show that $s=c+ps$. Solve this formula for s.

If the cost of a given article is to be 84 cents, by use of the last formula, find what must be the selling price in order to obtain a profit of 40 per cent.

12. A cistern contains 120 gal. of water. At a given moment water begins to flow into the cistern at the rate of a quarts per minute, and to flow out of another opening at the rate of b quarts per minute. Write a formula for the number of gallons (G) in the cistern at the end of t minutes. Also write the formula if the cistern contained G_0 gallons at the start instead of 120 gal.

13. If the proper height in inches of a chalk trough in a grade school room be denoted by h, and the number of the grade by G, the formula for the height of the chalk trough is $h=25+\frac{3}{2}(G-4)$. Convert this formula into a rule.

By use of the formula determine the proper height of the chalk trough in a room for the 6th grade. For your own grade.

14. If the length, width, and height of a room are denoted by l, w, h respectively, what does $2(l+w)h$ represent? Draw a diagram to show this.

15. Also what does $2(l+w)h+lw$ represent? Draw a diagram to show this.

16. Also what does lwh represent? Draw a diagram to show this.

17. Using suitable letters, construct a formula for the weight of a wagon and its load, if the load consists of two sets of objects, as a number of barrels of equal weight, and a number of boxes also of equal weight.

Illustrate the use of your formula by a numerical example.

18. What temperature is numerically the same on both the Centigrade and Fahrenheit scales?

SUG. Eliminate C between the equations $C=\frac{5}{9}(F-32)$ and $C=F$.

19. Find that temperature on the Fahrenheit scale which is numerically double the equivalent temperature on the Centigrade scale.

20. Also find that temperature on the Fahrenheit scale which is numerically one half the equivalent temperature on the Centigrade scale.

21. How many square miles are in a tract of land which contains 20,000 acres? If M means number of square miles and t means number of thousands of acres, find a formula for M in terms of t.

33. A Variable is a quantity which has an indefinite number of different values.

A **function** is a variable which depends on another variable for its value.

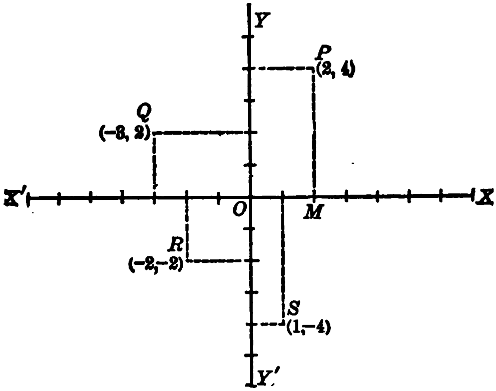
Thus, the *area* of a circle is a function of the *radius* of the circle; the *wages* which a laborer receives are a function of the *time* that the man works.

A **graph** is a diagram representing the relation between a function and the variable on which the function depends for its value.

A function may depend for its value on more than one variable. Thus, the area of a rectangle depends on two quantities—the length of the rectangle and the breadth. The present treatment of graphs, however, is limited to functions which depend on a single variable.

34. Uses of Graphs. A graph is useful in showing at a glance the place where the function represented has the greatest or least value and where it is changing its value most rapidly, and in making evident similar properties of the function.

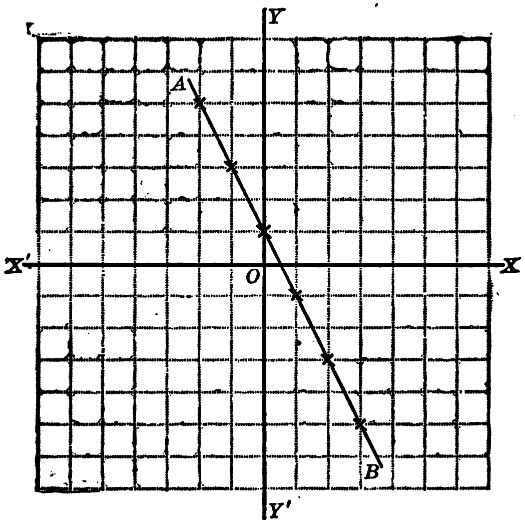
Graphs of algebraic equations are useful in making evident certain properties of equations which are otherwise difficult to understand. A graph also often furnishes a rapid method of determining the root (or roots) of an equation.



35. Location of Points. In the diagram above, the lines XX' and YY' are perpendicular to each other. If, in constructing graphs, the point P is considered as located with reference to XX' and YY', what are these lines called?

What name is given to the point O when the lines intersect? What to PM? To OM? Which are the four quadrants in the figure? Name them in order. Draw a similar set of axes on squared paper, and with reference to them locate the points $(3, 2)$, $(-5, 4)$, $(-3, -6)$, $(4, -2)$, $(3, 0)$, $(0, -5)$, $(0, 0)$.

36. To Construct the Graph of an Equation of the First Degree Containing Two Unknown Quantities, as x and y.



Let x have a series of convenient values, as 0, 1, 2, 3, etc., -1, -2, -3, etc.;

Find the corresponding values of y;

Locate the points thus determined, and draw a line through these points.

Ex. 1. Construct the graph of the equation $y=1-2x$.

x	y
0	1
1	-1
2	-3
3	-5
etc.	etc.
-1	3
-2	5
etc.	etc.

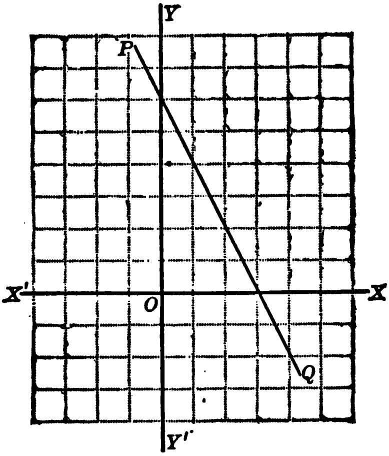
Construct the points (0, 1), (1, -1), (2, -3), (3, -5), (-1, 3), (-2, 5), etc., and draw a line through them. The straight line AB is thus found to be the graph of $y=1-2x$.

Ex. 2. Graph $y = 600 - 4x$.

x	y
0	600
50	400
100	200
-50	800

Let each space on the y-axis represent 100, and each space on the x-axis represent 50.

We then obtain PQ as the required graph.

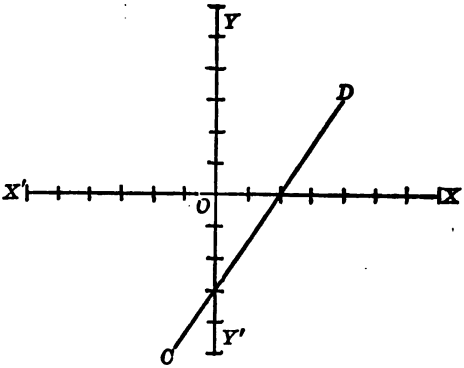


37. Linear Equations. It will always be found that the graph of an equation of the first degree which contains not more than two unknown quantities is a straight line. Hence,

A linear equation is an equation of the first degree.

38. Abbreviated Method of Constructing the Graph of a Linear Equation. Since a straight line is determined by two points, in order to construct the graph of an equation of the first degree, it is sufficient to *construct any two points of the graph and draw a straight line through them.*

Ex. 1. Graph $3x-2y=6$.



When $x=0, y=-3$; when $y=0$, $x=2$. Hence, the graph passes through the points $(0, -3)$ and $(2, 0)$, or *CD* is the required graph.

The greater the distance between the points chosen, the more accurate the construction will be. It is usually advisable to test the result obtained by locating a third point and observing whether it falls upon the graph as constructed.

If the given line does not pass through the origin, or near the origin on both axes, it is usually best to construct the line by determining the points where the line crosses the axes as was done in the above example.

EXERCISE 47

1. Draw axes and locate the points $(-4, 3.5)$, $(-2.6, 0)$, $(0, 3.2)$, $(-1.8, -1.2)$.

2. A point moves with reference to two axes so that its ordinate is always 3. What is the locus of the point?

3. A point moves with reference to two axes so that its abscissa is always 2. What is the locus of the point?

Graph the following:

4. $y=x-3$.
5. $3x+4y=12$.
6. $3x+2y=9$.
7. $2y=x$.
8. $.3x+.5y=1$.
9. $2x+3y=600$.
10. $200x+y=600$.
11. $\frac{x-3}{2}=\frac{1}{2}y$.
12. $x=3$.
13. $y=4$.
14. $y-3.5 \cdot x=2.4$.

15. $x=2(y-1)$.

16. $x=400-3x$.

17. $x=0$.

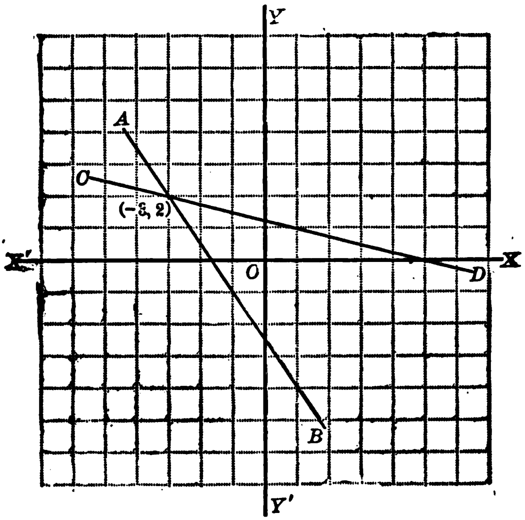
18. $y=0$.

19. Construct the graph of $M=\frac{25}{16}t$, the vertical axis being that of M. By use of this graph find the value of M when $t=10$.

What is the meaning of this problem as applied to the formula obtained in Ex. 21, p. 87?

20. By use of the graph obtained in Ex. 19, find t when $M=8$. What is the meaning of this example in relation to Ex. 21, p. 87?

21. In graphing $\frac{x-3}{2}=\frac{1}{2}y$, why is it an advantage to solve for y before substituting successive values of x in the equation?



39. Graphic Solution of Simultaneous Linear Equations. If we construct the graph of the equation $2y+3x=-5$ (the line AB) and the graph of $4y+x=5$ (the line CD), and measure the coördinates of their point of intersection, we find this point to be $(-3, 2)$.

If we solve the pair of simultaneous equations $\begin{cases} 2y+3x=-5 \\ 4y+x=5 \end{cases}$ by the ordinary algebraic method, we find that $x=-3$ and $y=2$.

In general, *the roots of two simultaneous linear equations correspond to the coördinates of the point of intersection of their graphs;* for these coördinates are the only ones which satisfy both graphs, and their values are also the only values of x and y which satisfy both equations.

Hence, to obtain the graphic solution of two simultaneous equations,

Draw the graphs of the given equations, and measure the coördinates of the point (or points) of intersection.

Graphing two simultaneous equations is a convenient method of testing or checking their algebraic solution.

40. Simultaneous Linear Equations whose Graphs are Parallel Lines. Construct the graph of $3x-2y=6$ and also of $3x-2y=2$.

You will find that the graphs obtained are parallel straight lines. Now try to solve the same equations algebraically. You will find that when either x or y is eliminated, the other unknown quantity is eliminated also, and that it is therefore impossible to obtain a solution. The reason why an algebraic solution is impossible is made clear by the fact that the graphs, being parallel lines, cannot intersect; that is to say, there are no values of x and y which will satisfy both of these lines, or both equations, at the same time.

41. Graphic Solution of an Equation of the First Degree of One Unknown Quantity. By substituting for y in the first equation of the pair $\begin{cases} y=2x-5, \\ y=0, \end{cases}$ the two equations reduce to $2x-5=0$. Accordingly, the graphic solution of an equation like $2x-5=0$ can be obtained by combining the graphs of $y=2x-5$ and $y=0$. In other words, the root of $2x-5=0$ is represented graphically by the abscissa of the point where the graph of $y=2x-5$ crosses the x-axis.

EXERCISE 48

Solve both algebraically and graphically:

1. $2x-7y=9$.
   $5x+3y=2$.
2. $3x-2y=1$.
   $y=2-x$.
3. $x+3+5y=0$.
   $7x+8y=6$.
4. $x=3$.
   $2y+3x=5$.
5. $3x+2y=21$.
   $y=2x$.
6. $x-\frac{y-2}{7}=5$.
   $4y=3+\frac{x+10}{3}$.
7. $6x-5y=3$.
   $5x-6y=8$.
8. $y=3x+9$.
   $2x+7y+6=0$.
9. $2y-x=2$.
   $y=2$.
10. $x-4y=6$.
    $y+1=0$.
11. $y=0$.
    $y=2x+3$.
12. Solve graphically $2x+3=0$.
13. Solve graphically $3x-5=0$.

14. Draw the graphs of $2y+3x+6=0$ and $4x-y+5=0$ and on the diagram estimate to the nearest tenth the values of x and y at the point of intersection of the graphs.

15. Solve graphically to the nearest tenth $\begin{cases} 8x+5y=9. \\ 6x-2y=11. \end{cases}$

16. Construct the graphs of $\begin{cases} 3x-2y=4. \\ 9x-6y=4. \end{cases}$

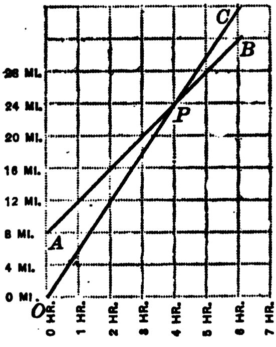
Can you solve this pair of equations algebraically? Give reasons for your answer.

17. Construct the triangle whose sides are the graphs of the equations $3y+8x=26$, $4y-9x=15$, $7y-x+18=0$, and find the coördinates of the vertices of the triangle.

18. Construct the quadrilateral whose sides are the graphs of the equations $y+2x=10$, $5y+2x=26$, $y+11x+16=0$, $6y-5x+25=0$, and find the coördinates of the vertices of the quadrilateral.

42. Graphical Solution of Problems.

Ex. 1. A man starts to walk toward a certain place at the rate of 4 miles an hour. After he has gone 8 miles, a boy on a bicycle starts after him and goes at the rate of 6 miles an hour. Show graphically how many hours and how many miles the boy will have to travel in order to overtake the man.



On the diagram each space on the horizontal axis represents 1 hour, each space on the vertical axis represents four miles. Then the line *OC* represents the relation between the number of miles and number of hours traveled by the boy and *AB* in like manner represents the motion of the man after the boy starts.

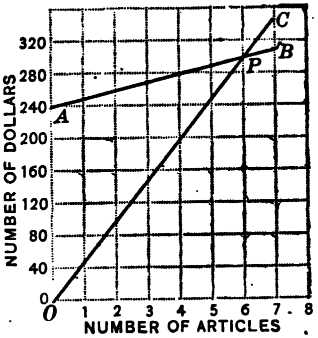
Hence, the time the boy travels is 4 hours, and the distance he travels is 24 miles.

The point *P*, where the two graphs intersect, determines the number of hours and miles traveled before the boy overtakes the man.

These results may be checked algebraically by noting that if d denotes distance (or number of miles), and t time (or number of hours) after the boy starts, the equation for the distance traveled by the man is $d=8+4t$, and that for the boy $d=6t$. Solving these equations, $d=24$, and $t=4$.

In other words, *AB* is the graph of the equation $d=8+4t$, and *OC* is the graph of the equation $d=6t$.

Ex. 2. The cost of making a certain article by hand is \$50 each. If the article be made by machinery, the of preparing the machinery will be \$240, after which each article can be made for \$10. How many articles must be made before the use of machinery becomes profitable?



Let each space on the vertical axis represent \$40. Then the line *AB* (graph of formula $c=240+10\,n$) represents the cost of the machine-made article, and *OC* (graph of the formula $c=50\,n$) represents the cost of the hand-made article.

The two lines intersect at *P*. Hence, 6 articles must be made before the use of machinery becomes profitable.

EXERCISE 49

Solve the following graphically and check by the use of formulas:

1. A boy started to walk toward a certain place at the rate of 3 miles an hour. After he had gone 9 miles, another boy on a bicycle started after him and went at the rate of 6 miles an hour. How many hours and how many miles must the second boy go in order to overtake the first?

2. A freight train started from a certain station and proceeded at the rate of 20 miles an hour. After it had gone 30 miles, it was followed by a passenger train going 40 miles an hour. How many hours and miles will the passenger train have traveled before it overtakes the freight train?

3. The cost of manufacturing a certain article by hand is \$10 each. If the article be made by machinery, the cost of preparing the machinery is \$180, after which the cost of making each article is \$4. How many articles

must be made by machinery before such manufacture becomes profitable?

4. A merchant wishes to send a circular letter to a number of his customers. He finds that to prepare typewritten copies will cost 5 cents a copy, but that to have the letter printed will cost $5 for the first hundred copies, and $1 for each hundred after that. What is the least number of letters he must send to make it pay to have the letters printed?

SUG. Observe that the cost of preparing to have the letters printed is $4, that is, $5 less $1 (the cost of printing the first hundred copies). Hence the cost of printing the letters may be stated as $4 plus $1 for each hundred copies.

5. At a certain place it costs 20 cents each to hew railroad ties by hand. After spending $1000 for machinery, they could be sawed for 4 cents each. How many would have to be sawed before it became profitable to use machinery?

6. After spending $600 in getting special training, a youth obtained a position which paid a weekly wage of $10 with an increase which amounted to $2 a week. At the same time a friend, who had begun working earlier without waiting to get an education, was receiving $15 a week with an increase of 75 cents a week. How many weeks must elapse before the weekly wages of the educated youth will equal that of the other? When will it be $12 greater?

7. A boy starts from a certain place and travels at the rate of 8 miles an hour. Three hours after he starts, another boy goes after him on a motorcycle and travels at the rate of 32 miles an hour. How many miles and how many hours must the second boy travel in order to overtake the first?

CHAPTER V

INVOLUTION AND EVOLUTION

INVOLUTION

43. Involution is the operation of raising an expression to any required power.

Since a power is the product of equal factors, involution is a species of multiplication. In this multiplication, the fact that the quantities multiplied are equal leads to important abbreviations of the work.

44. Law of Exponents or Index Law.

Since $a^3 = a \times a \times a,$

$$(a^3)^4 = (a \times a \times a)(a \times a \times a)(a \times a \times a)(a \times a \times a)$$

$$= a^{3 \times 4} = a^{12}.$$

In general, in raising a^n to the m^{th} power, we have the factor a taken $m \times n$ times, or

$$(a^n)^m = a^{mn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{I.}$$

Also, $(ab)^n = ab \times ab \times ab \times ab,$ . . to n factors

$$= (a \times a \times a \ \ldots \text{ to } n \text{ factors})(b \times b \times b \ \ldots \text{ to } n \text{ factors})$$

$$\therefore \ (ab)^n = a^n b^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{II.}$$

This law enables us to reduce the process of finding the power of a product to the simpler process of finding the power of each factor of the given product.

45. Law of Signs. It is evident from the law of signs in multiplication that

(1) *An even power of a quantity (whether plus or minus) is always positive.*

Thus, $(-5\,ab)^4 = 625\,a^4b^4.$

(2) *An odd power of a quantity has the same sign as the original quantity.*

Thus, $(-2\,a)^7 = -128\,a^7.$

46. Involution of Monomials in General. Hence, to raise a monomial to a required power,

Raise the coefficient to the required power;

Multiply the exponent of each literal factor by the index of the required power;

Prefix the proper sign to the result.

Ex. $(-5\,a^2x^3)^4 = 625\,a^8x^{12}.$ *Ans.*

47. Powers of Fractions. By a method similar to that used in § 44, it can be shown that

$$\left(\frac{a^n}{b^m}\right)^p = \frac{a^{np}}{b^{mp}}.$$

Hence, to raise a fraction to a required power,

Raise both numerator and denominator to the required power, and prefix the proper sign to the resulting fraction.

Ex. $\left(-\frac{3\,ab^2}{5\,x^2y^3}\right)^3 = -\frac{27\,a^3b^6}{125\,x^6y^9}.$ *Ans.*

EXERCISE 50

Write the square of

1. $7\,a^2b.$
2. $-\frac{2}{3}\,x^2y^5.$
3. $-13\,x^ny.$
4. $-4\,y^{n-2}.$
5. $-\frac{3\,x}{5\,z^2}.$
6. $\frac{x^ny^m}{10\,z}.$

Write the cube of

7. $3x^2y^3$.

8. $-5x^ny$.

9. $-\frac{3x}{4y^2}$.

10. $\frac{5cd^5}{7x^{2n}}$.

11. $\frac{a^nb^n}{c^{n-2}}$.

Write the value of

12. $(7ab^2c^3)^3$.

13. $(\frac{3}{2}x^2y)^4$.

14. $\left(\frac{2x^2y^2}{a^nz}\right)^5$.

15. $(-2x^3)^7$.

16. $(-\frac{1}{2}m^2)^6$.

17. $(3\frac{1}{2}x^{2n})^3$.

18. $(.003)^3$.

19. $\left(-\frac{5x^n}{3a^{n-1}}\right)^4$.

20. $\frac{1}{2}\left(\frac{2}{3}\right)^5$.

21. The number 5,000,000,000 may be expressed in a brief form by the use of an exponent thus, 5×10^9. Similarly, express 19,000,000,000,000 in a brief form. Compare the number of figures in the long and short forms.

22. The nearest fixed star is approximately 20 millions of millions of miles from the earth. Express this number in a brief form by the aid of an exponent.

23. Give the value of 2×3^2, $(2\times3)^2$.

24. Give the value of each of the following: $[(-2)^3]^2$, $[(-2)^2]^3$, $[(-3)^2]^2$. Does $[(-a)^3]^2$ equal $[(-a)^2]^3$?

25. Does 2×3^5 equal 6^5? Does $\frac{1}{2}(4^5)$ equal 2^5?

26. Given $2^5=32$, find in the shortest way the value of 2^{10}. Also of 2^{15}. Of 2^{12}.

27. In $x^3\cdot x^5$, how many x's are multiplied together? How many in $(x^3)^5$? Write out in full each of these set of factors.

28. State how many x's are multiplied together in $[(x^2)^3]^4$. Also in $(x^2)^3\cdot x^4$.

29. In reducing $\frac{(x^3)^5}{x^3\cdot x^5}$ to its simplest form, is it allowable to cancel the 5's? Why? Is it allowable to cancel the 3's?

30. Does $2^3\times2^4$ equal 4^7? Give a reason for your answer. The value of the second expression is how many times as great as that of the first?

31. Express as a power of 2 the number of great-grandparents a person has. Also the number of great-great-great-grandparents.

32. If a decimal fraction contains four places, how many places will its square contain? Its cube? Its fourth power? Give a numerical illustration of your answer to the first of these questions.

33. Show that $2^8 \times 5^8 = 10^8$. Is there any advantage in knowing that $2^8 \times 5^8 = 10^8$?

48. A Root of a given quantity is a quantity which, taken as a factor a certain number of times, will produce the given quantity.

EVOLUTION

49. Evolution is the process of finding a required root of a quantity.

What is the *radical* or root sign? What is the meaning of $\sqrt{9}$? Of $\sqrt[3]{a}$? Of $\sqrt{7}$?

50. Number of Roots. Taking a particular example, we find that $\sqrt{4}$ has two values, viz.: $+2$ and -2, for $(+2)^2=4$ and $(-2)^2=4$.

A number containing a square root of a negative quantity is termed an **imaginary number.**

A **real number** is a number which does not contain an imaginary number.

The nature of the square of an imaginary number, as of $\sqrt{-4}$, is explained on page 136.

If we include imaginary roots, it may be shown that when any root of a given number is extracted, *the number of possible roots equals the index of the root to be extracted.*

Thus, in taking the cube root of 8, we find three possible roots; viz., 2, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$.

51. The Principal Root of a number is that real root of the number which has the same sign as the number itself.

Thus, the principal root for $\sqrt{4}$ is 2; for $\sqrt[3]{27}$ is 3; for $\sqrt[3]{-27}$ is -3.

In this chapter, only the principal roots of numbers are considered.

52. Index Law. Since $(a^m)^n = a^{mn}$ (§ 44, p. 98), it follows that

$$\sqrt[n]{a^{mn}} = a^m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{I.}$$

where m and n are positive integers.

Hence, finding the root of a quantity affected by an exponent becomes simply a division of exponents.

Also, $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$. . . . . II.

For, let $\sqrt[n]{a} = x, \quad \sqrt[n]{b} = y;$

$\therefore \; x^n = a.$ . . . (1) $\qquad y^n = b.$ . . . . (2)

But $x^n y^n = (xy)^n$ (by § 44).

Substituting for x^n and y^n from (1) and (2),

$$ab = (\sqrt[n]{a}\sqrt[n]{b})^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Extracting the nth root of each member of (3),

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}.$$

This reduces the process of finding the nth root of a product to the simpler process of finding the nth root of each of the factors composing the product.

53. Method. Hence, to extract a required root of any monomial,

Extract the required root of the coefficient;

Divide the exponent of each letter by the index of the required root;

Prefix the proper sign to the result.

How may the work be checked?

EXERCISE 51

1. Write the square root of $16\,a^2x^6$, $9\,x^6y^8$, $\frac{4}{9}\,a^2x^2y^4$, $\frac{9}{16}\,a^{4n}$.

Write the value of

2. $\sqrt{81\,a^2b^4}$.
5. $\sqrt[4]{16\,a^4b^8}$.
8. $\sqrt[4]{\frac{81}{16}\,a^4y^{12}}$.
3. $\sqrt[3]{27\,a^6b^6}$.
6. $\sqrt[5]{32\,a^5x^{10}}$.
9. $\sqrt[5]{-\frac{32\,a^{10}}{243\,b^{5n}}}$.
4. $\sqrt[3]{-27\,a^6b^6}$.
7. $\sqrt[6]{64\,a^{12}x^6}$.

Find the value of x in each of the following equations:

10. $x^4=81$.
11. $x^7=128$.
12. $x^5=-32$.

54. Square Root of Polynomials. In squaring a trinomial, $a+b+c$, we may regard $a+b$ as a single quantity, and denote it by a symbol, as p. We then obtain the square in the form $p^2+2\,pc+c^2$.

Evidently we may reverse this process, and extract a square root to three terms by regarding two terms of the root, when found, as a single quantity. Similarly, a fourth term of a root, or any number of terms, may be obtained by regarding the root already found as a single quantity.

Ex. Extract the square root of $a^4+10\,a^3b+19\,a^2b^2-30\,ab^3+9\,b^4$.

$$\begin{array}{l|l}
 & a^4+10\,a^3b+19\,a^2b^2-30\,ab^3+9\,b^4 \;\big|\; \underline{a^2+5\,ab-3\,b^2}.\quad \textit{Root.} \\
 & a^4 \\
\hline
2\,a^2+5\,ab & 10\,a^3b+19\,a^2b^2 \\
 & 10\,a^3b+25\,a^2b^2 \\
\hline
2\,a^2+10\,ab-3\,b^2 & -6\,a^2b^2-30\,ab^3+9\,b^4 \\
 & \underline{-6\,a^2b^2-30\,ab^3+9\,b^4}
\end{array}$$

Check the work by squaring the result obtained, or by numerical substitution. Let the pupil state the above process as a rule.

55. Square Root of Arithmetical Numbers. The same general method as that used in § 54 may be used to extract the square root of arithmetical numbers.

The details of the method of extracting the square root of numbers are explained in arithmetic (see *Durell's Advanced Arithmetic*). As an illustration of the process, we give the following example:

Ex. Extract the square root of 679.6449.

$$\begin{array}{r|l}
 & \overset{\frown}{6}\overset{\frown}{79}.\overset{\frown}{64}\overset{\frown}{49} \;\big|\; \underline{26.07.} \quad \textit{Root.} \\
 & 4 \\ \hline
46 & 279 \\
 & 276 \\ \hline
5207 & 36449 \\
 & 36449 \\ \hline
\end{array}$$

EXERCISE 52

Extract the square root and check:

1. $29\,b^2-26\,b^4+12\,b^5-14\,b^3-10\,b+9\,b^6+1.$
2. $4\,p^6-12\,b^2p^3+20\,p^3+9\,b^4-30\,b^2+25.$
3. $a^2-2\,ab+b^2+25+10\,a-10\,b.$
4. $24\,a^5b+24\,ab^5+9\,a^6+9\,b^6-8\,a^4b^2-8\,a^2b^4-50\,a^3b^3.$
5. $22\,x^4-20\,x^3+4-4\,x+17\,x^2+9\,x^6-24\,x^5.$
6. $x^4-2\,x^3+\frac{3\,x^2}{2}-\frac{x}{2}+\frac{1}{16}.$
7. $a^4+\frac{b^4}{4}+\frac{ab^3}{3}-\frac{2\,a^3b}{3}-\frac{8\,a^2b^2}{9}.$
8. $4\,p^2-\frac{3}{2}\,p-4\,p^3+\frac{9}{16}+4\,p^4.$

Find the square root of

9. 283024.
10. 9312.25.
11. 8042896.
12. 4916.8144.
13. .64048009.
14. 10.06475625.

Find the square root to nearest thousandth:

15. 17.5 **17.** .4. **19.** $\frac{7}{13}$. **21.** 9.0042.
16. $7\frac{2}{3}$. **18.** $9\frac{7}{12}$. **20.** .081. **22.** 176.5.

23. In extracting the square root of $\frac{2}{3}$, what is the advantage of reducing $\frac{2}{3}$ to a decimal fraction before extracting any root?

Compute to three places the value of

24. $\sqrt{3+\sqrt{2}}$. **25.** $\sqrt{4\sqrt{7}-\sqrt{5}}$. **26.** $\sqrt{\frac{3(\sqrt{7}+\sqrt{2})}{2}}$.

27. Find the altitude of an equilateral triangle whose side is 27 in.

28. Find the side of an equilateral triangle whose altitude is 27 in.

29. Find the side of a square whose diagonal is 36 in.

30. If a city park is 800 yd. long and 600 yd. wide, how much is saved by walking from a corner to the opposite corner along a diagonal instead of along the sides? How much time does a man save in a year by walking thus, if he walks at the rate of 4 miles an hour, and crosses the park four times a day on 300 days?

31. If $K=\sqrt{s(s-a)(s-b)(s-c)}$, $a=126$, $b=50$, $c=148$, $s=\frac{1}{2}(a+b+c)$, find K. What is the meaning of this process in geometry?

32. Find in feet the radius of a circle whose area is one square rod.

33. Find in feet the length of the tether by which a cow must be tied in order that she may graze over two fifths of an acre?

34. If $4\,\pi R^2=200$ sq. in., and $\pi=\frac{22}{7}$, find R. What is the meaning of this process as applied to the sphere?

35. The area of California is 158,300 sq. mi. Find the side of a square having an equivalent area. How may you then visualize the area of California?

36. By the use of the method of Ex. 35, visualize the area of the state in which you live.

37. A bushel measure is to be square and 12 in. deep. Find in inches the inside edge of the square top.

38. The inside dimensions of the bottom of a trunk are 27 in. and 39 in. What is the length of the longest umbrella which can lie flat on the bottom of the trunk?

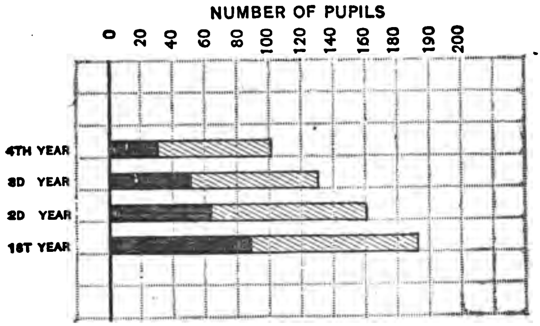
39. What is the diameter of the largest wheel that can be made to pass through a door 40 in. wide and 7 ft. high?

56. Dissected Bar Graphs.

Ex. The enrollment of boys and girls in a certain high school by grades is given in the adjoining tabulation. Make a graphical representation of these numbers by the use of bars divided into parts.

	Boys	Girls
1st year	88	102
2d year	64	97
3d year	48	82
4th year	30	70

The graphical representation is given in the diagram below. In each bar the left-hand or black part represents a number of boys, and the right-hand part represents the corresponding number of girls.



EXERCISE 53

Make dissected bar graphs for the following:

1. In a certain college the numbers of men and women students in the different classes were as follows:

	MEN	WOMEN
Freshman	432	316
Sophomore	402	256
Junior	386	203
Senior	354	122

2. The parts of the cotton crop of the United States (in millions of bales) used at home and abroad in the years specified were as follows:

YEAR	1870	1880	1890	1900	1910	1920
Used at home..........	1.1	1.9	2.4	3.5	4.3	
Exported.............	2	3.8	4.9	5.9	6.8	

3. For three different railroads the per cents that the passenger and freight receipts were of the total receipts were as follows in a given year:

	PASSENGER	FREIGHT
1st railroad	68%	32%
2d railroad	37%	63%
3d railroad	46%	54%

4. In a certain year in two different states, the per cents of native white, foreign-born white, and colored population were as follows: (Dissect the bar for each state into three parts.)

	NATIVE WHITE	FOREIGN WHITE	COLORED
1st state	88	10	2
2d state	56	4	40

5. In two successive years the sales of a company were as follows: (Dissect each bar into four parts.)

	COAL	LUMBER	LIME	FERTILIZERS
1st year	$24,500	$32,800	$6,500	$12,010
2d year	28,700	22,700	7,610	13,216

6. In a given year in four different states in the United States, the per cents of literates and illiterates were as follows:

	LITERATES	ILLITERATES
1st state	98.3%	1.7%
2d state	94	6
3d state	88	12
4th state	71	29

Make circle graphs for the data in Exs. 7, 8, and 9.

7. The per cents of the population formed by the different races of the world are as follows: Yellow, 45; white, 41; black, 11; brown, 2; red, 1.

8. Of 100 tons of coal burned in locomotives, the following are the per cents utilized and lost or wasted in various ways, on the average: Utilized in boiler, 41; lost through radiation and leakage, 23; lost in gases in smoke-stacks, 10; unconsumed in ashes, 12; lost in other ways, 14.

Represent these figures also as a dissected bar.

9. A certain man distributed the 24 hours of an ordinary day as follows: sleep, 8; hard work, 6; light work, 4; meals, 2; recreation, 4.

Represent these figures also as a dissected bar.

EXERCISE 54

REVIEW

Solve and check:

1. The length, width, and height of a room in feet are l, w, h, respectively. The walls of the room contain n windows each $b \times c$ inches and d doors each $e \times f$ inches. Obtain a formula for the cost (c) in dollars of lathing and plastering the walls and ceiling of the room at p cents per square yard, deducting one half of the door and window space.

Make up a numerical example and solve it by use of this formula.

2. Solve for x and check:

$$\tfrac{1}{3}(x-a)-\tfrac{1}{5}(2x-3b)-\tfrac{1}{2}(a-x)=10a+11b.$$

3. Factor:

(1) $a(4a+4b)+b^2$.
(3) $2a-2b+ax-bx$.
(2) $.0016-x^4$.
(4) $x^{12}-1$.
(5) $x^4+11x^3+41x^2+61x+30$.

Graph the numerical facts given in Exs. 4, 5, 6, in each case selecting the most appropriate form of graph.

4. The numbers of subscribers' stations on the Bell telephone system and its connections in the United States were as follows in the years specified (in graphing let each space on the vertical axis represent 500,000).

YEAR	1880	1885	1890	1895	1900	1905	1910	1915	1920
No. of thousands of stations........	31	148	212	270	677	2005	5143	8649	11,796

5. The per cents of proteins in six important foods are as follows: Oatmeal, 16.1; whole wheat flour, 13.8; white flour, 11.4; corn meal, 9.2; rice, 7.8; potatoes, 2.2.

6. A certain business disposed of its total revenue for a year as follows: Salaries and wages, 50 %; purchase of materials, 22 %; dividends, 12 %; taxes, 5 %; surplus, 5 %; insurance, 3 %; miscellaneous expenses, 3%.

7. Using suitable letters, make a formula for the entire weight of a wagon and its load, which consists of three sets of like objects (see Ex. 17, p. 86).

8. Extract the square root of $x^4+.4x^3-.96x^2-.2x+.25$.

9. A half-mile running track is to have the shape shown in Ex. 4, p. 85. If the radius of the semi-circular ends is to be 200 ft., by use of the formula obtained in Ex. 4, p. 85, find the length of the straight sides of the track.

10. Eliminate p between $a=bp+c$ and $d=lp+f$.

11. Compute the numerical value of $\sqrt{5.2+2\sqrt{3.72}}$ to the nearest hundredth.

12. Solve $x+3y+3z=1$, $3x-5z=1$, $9y+10z+3x=1$, and check.

13. Simplify $\frac{4\,ax}{3\,by}\cdot\frac{a^2-x^2}{c^2-x^2}\div\frac{a^2-ax}{bc+bx}$ and check by letting $a=1$, $b=2$, $c=3$, $x=4$, and $y=5$.

14. How much milk containing 3.4 % of butter fat must be mixed with how much cream containing 26.5 % of butter fat to make a mixture of 100 gal. containing 12 % of butter fat?

15. Solve $r=\frac{b}{l^2}\left(\frac{1}{c}+d\right)$ for c.

16. Solve graphically to the nearest tenth $3\,x+5\,y=8$ and $4\,x-3\,y=12$.

17. How much pure alcohol must be added to 2 gallons of a mixture which is 80 % pure to make a mixture which is 90 % pure?

18. Divide both sides of the formula $v=lwh$ by lw. What is the meaning of the result?

19. The following table gives the population (in millions) of the British Isles, France, Germany, and the United States in the years specified.

YEAR	1800	1810	1820	1830	1840	1850	1860	1870	1880	1890	1900	1910	1920
British Isles	16	18	21	24	26	27	29	31	35	38	41	45	
France	27	29	31	32	34	36	37	36	38	39	39	40	
Germany	22	23	26	30	32	35	38	41	45	49	56	65	
United States	5	7	10	13	17	23	31	39	50	63	76	92	

Graph these four sets of figures on one chart. From the chart determine as nearly as you can the year in which the populations of the following countries were equal: (1) British Isles and United States; (2) France and Germany; (3) British Isles and France; (4) British Isles and Germany.

20. A man has an income of i dollars. All of this above $2000 is subject to a tax of a per cent (expressed decimally), and all above $5000 is subject to an additional tax of b per cent expressed decimally. Denoting the total tax by T, obtain a formula for T.

By use of this formula, find the income tax of a man whose income is $5800 when $a=4$ % and $b=2$ %.

CHAPTER VI

EXPONENTS; RADICALS; IMAGINARIES

57. Positive Integral Exponents. Using a^3 as a brief symbol for $a \times a \times a$, and a^m as a brief symbol for $a \times a \times a \times a$ . . . to m factors, we have already found the following laws to govern the use of positive integral exponents:

I. $a^m \times a^n = a^{m+n}$.

II. $\dfrac{a^m}{a^n} = a^{m-n}$, if $m > n$.

III. $(a^m)^n = a^{mn}$.

IV. $\sqrt[n]{a^{mn}} = a^m$.

V. $(ab)^n = a^n b^n$.

We shall now find that it is both possible and desirable to regard these laws as true when m and n are fractional and negative numbers.

58. I. Meaning of a Fractional Exponent. By addition of exponents, $a^2 \times a^2 \times a^2 = a^{2+2+2} = a^6$.

If we apply the same method to fractional exponents, we get, for example,

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^{\frac{6}{3}} = a^2.$$

that is, $a^{\frac{2}{3}}$ may be regarded as one of three equal factors composing a^2; that is, $a^{\frac{2}{3}}$ is the cube root of a^2.

$$\therefore\ a^{\frac{2}{3}} = \sqrt[3]{a^2}.$$

So, in general,

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \text{ . . . to } q \text{ factors}$$

$$= a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\dots \text{ to } q \text{ terms}}$$

$$= a^{\frac{p}{q}\times q} = a^p.$$

Hence, in general, *in a fractional exponent the numerator denotes the power of the base that is to be taken, and the denominator denotes the root that is to be extracted.*

Ex. 1. $27^{\frac{4}{3}}=\sqrt[3]{27^4}=3^4=81.$ *Ans.*

Ex. 2. $a^{\frac{5}{4}}\times a^{\frac{2}{3}}\times a^{\frac{1}{2}}=a^{\frac{5}{4}+\frac{2}{3}+\frac{1}{2}}=a^{\frac{29}{12}}.$ *Ans.*

Ex. 3. $\sqrt{2}\,x^{a+b}\cdot 2^{\frac{3}{2}}x^{a-b}=2^{\frac{1}{2}}x^{a+b}\cdot 2^{\frac{3}{2}}x^{a-b}=2^2x^{2a}=4\,x^{2a}.$ *Ans.*

Note that in Ex. 1 it is best to *extract the required root first.*

In the examples which involve letters, the work may often be checked by numerical substitutions.

EXERCISE 55

Express with radical signs:

1. $x^{\frac{3}{2}}$. **3.** $3\,b^{\frac{1}{4}}$. **5.** $5\,x^{\frac{1}{2}}y^{\frac{1}{4}}$. **7.** $5\,x^{\frac{n}{3}}$.

2. $x^{\frac{5}{6}}$. **4.** $2\,a^{\frac{3}{2}}b^{\frac{5}{4}}$. **6.** $3\,a^2x^{\frac{4}{3}}$. **8.** $ab^{\frac{1}{2}}y^{\frac{n-2}{n}}$.

Express with fractional exponents:

9. $\sqrt[3]{b^5}$. **11.** $5\sqrt[3]{y^4}$. **13.** $\sqrt[3]{x^2}\sqrt[5]{y^4}$. **15.** $\dfrac{3\,x\sqrt[3]{y}}{ab^{\frac{1}{2}}\sqrt[5]{c}}$.

10. $\sqrt{x}$. **12.** $a\sqrt[5]{x}$. **14.** $3\sqrt{a}\sqrt[3]{y}$.

Find the value of

16. $9^{\frac{3}{2}}$. **20.** $\sqrt{25^3}$. **24.** $(-125)^{\frac{2}{3}}$. **28.** $(.25)^{\frac{3}{2}}$.

17. $16^{\frac{5}{4}}$. **21.** $\sqrt[4]{16^3}$. **25.** $(\frac{9}{16})^{\frac{3}{2}}$. **29.** $(.36)^{\frac{1}{2}}$.

18. $81^{\frac{3}{4}}$. **22.** $(-27)^{\frac{2}{3}}$. **26.** $(-\frac{8}{27})^{\frac{4}{3}}$. **30.** $(.125)^{\frac{2}{3}}$

19. $\sqrt[3]{27^2}$. **23.** $(-32)^{\frac{3}{5}}$. **27.** $(\frac{16}{81})^{\frac{5}{4}}$. **31.** $(.008)^{\frac{2}{3}}$.

Simplify by performing the operations indicated:

32. $a^{\frac{1}{2}}\times a^{\frac{3}{2}}$. **35.** $3^{\frac{1}{2}}a^{\frac{5}{3}}\times 3^{\frac{3}{2}}a^{\frac{1}{3}}$. **38.** $2^{\frac{1}{2}}\times 2^{\frac{5}{2}}$.

33. $3\,b^{\frac{1}{2}}\times 2\,b^{\frac{5}{3}}$. **36.** $a^{\frac{3}{2}}\times a^{\frac{2}{3}}$. **39.** $\sqrt[3]{5}\cdot\sqrt{5^3}$.

34. $a^{\frac{1}{4}}y^{\frac{3}{2}}\times a^{\frac{3}{4}}y^{\frac{3}{2}}$. **37.** $a^{\frac{4}{3}}\div a^{\frac{2}{3}}$. **40.** $b^{\frac{3}{5}}\sqrt{x^3}\cdot x^{\frac{1}{4}}$. i.

41. $3^{\frac{3}{2}} \cdot 3^{\frac{5}{4}} \cdot 3^{\frac{3}{4}}$.

42. $y^{a+b} \cdot y^{a-b}$.

43. $x^{3a-2b} \cdot x^{3a+2b}$.

44. $a^{2+p} \cdot a^{p-3} \cdot a^{3p-5}$.

45. $b^{3+m} \cdot b^{5-n} \div b^{6-n}$.

46. $\dfrac{3\,a^{\frac{2}{3}}\sqrt{x^5}}{5\sqrt{b}} \cdot \dfrac{4\sqrt[3]{a}\sqrt{x}}{3\sqrt{b^3}}$.

47. $\dfrac{\sqrt{3}\sqrt[5]{5}}{\sqrt[3]{2}\sqrt[4]{7}} \times \dfrac{3^{\frac{3}{2}}\sqrt[5]{5^4}}{2^{\frac{5}{3}}\sqrt[7]{7^{14}}}$.

48. $5^{1.672} \cdot 5^{2.153}$.

49. $a^{3.719} \div a^{1.253}$.

50. $10^{2.3674} \div 10^{1.5431}$.

51. Find the value of $5^{\frac{1}{2}}$ to three decimal places. (See § 55.) Also of $5^{\frac{3}{2}}$ or $\sqrt{125}$. Multiply the two results. Compare the amount of work in this combination of processes with that of finding the value of 5^2. Which method gives the more accurate result?

52. What two parts are there to every power? What is the difference between an exponent and a power?

59. II. Meaning of the Exponent Zero (or of a^0).

By direct division, $\dfrac{a^m}{a^m} = 1$.

By subtraction of exponents, $\dfrac{a^m}{a^m} = a^0$. $\therefore\ a^0 = 1$.

Thus, a^0 may be regarded as the result of dividing some power of a by itself.

An expression like $px^2 + qx + r$ is sometimes written $px^2 + qx + rx^0$, the advantage being that in the latter form every term contains an x.

60. III. Meaning of a Negative Exponent.

If we extend Law I of § 57 to cover negative exponents,

$$a^n \cdot a^{-n} = a^0 = 1.$$

$$\therefore\ a^{-n} = \frac{1}{a^n}.$$

Ex. 1. $2^{-3}=\frac{1}{2^3}=\frac{1}{8}.$ *Ans.*

Ex. 2. $4^{-\frac{3}{2}}=\frac{1}{4^{\frac{3}{2}}}=\frac{1}{8}.$ *Ans.*

Negative exponents are useful in enabling us to write certain decimal fractions in an abbreviated form.

Ex. 3. Express .000000007 in a briefer form by the use of negative exponents.

$$.000000007=\frac{7}{1,000,000,000}=\frac{7}{10^9}=7\times10^{-9}.\quad \textit{Ans.}$$

61. Transference of Factors in Terms of a Fraction. It follows from the meaning of a negative exponent that *any factor may be transferred from the numerator to the denominator of a fraction, or vice versa, provided the sign of the exponent of the factor is changed.*

Ex. 1. Transfer to the numerator the factors of the denominator of $\frac{5x}{a^{-1}b^2y^{-\frac{3}{2}}}.$

$$\frac{5x}{a^{-1}b^2y^{-\frac{3}{2}}}=5ab^{-2}xy^{\frac{3}{2}}.\quad \textit{Ans.}$$

Ex. 2. Express with positive exponents $\frac{5ax^{-3}}{3b^{-2}y^{-\frac{1}{2}}}.$

$$\frac{5ax^{-3}}{3b^{-2}y^{-\frac{1}{2}}}=\frac{5ab^2y^{\frac{1}{2}}}{3x^3}.\quad \textit{Ans.}$$

EXERCISE 56

Transfer to the numerator all the factors of the denominator in

1. $\frac{x^5}{a^2b^3}.$ 2. $\frac{5a}{xy^{-3}}.$ 3. $\frac{7a}{3x^5}.$ 4. $\frac{5}{4^{-1}a^{\frac{2}{3}}b^{-\frac{1}{2}}}.$

Express with positive exponents:

5. a^2b^{-3}. **7.** $5\,a^{-2}y^{-\frac{1}{2}}$. **9.** $3(-y)^{-2}$.

6. $\dfrac{x^{-3}y^5}{a^{-\frac{1}{2}}b^{-5}}$. **8.** $\dfrac{3\,a^{-3}b^{-\frac{1}{2}}}{4\,x^{-1}y^3}$. **10.** $a^{-1}+b^{-1}$.

Obtain the value of

11. $9^{-\frac{1}{2}}$.

12. $9^{-\frac{3}{2}}$.

13. $\dfrac{1}{\sqrt{4^{-5}}}$.

14. $\dfrac{1}{4^{-\frac{3}{2}}}$.

15. $\dfrac{8^{-\frac{2}{3}}}{4^{-2}}$.

16. $\dfrac{1^{-\frac{5}{3}}}{4^{-\frac{3}{2}}}$.

17. $4^{-2}\times 8^{-\frac{2}{3}}$.

18. $(.125)^{-\frac{2}{3}}$.

19. $(.0625)^{-\frac{1}{2}}$.

20. $\dfrac{3}{4^{-2}+3^{-2}}$.

21. $(12\frac{1}{4})^{-\frac{3}{2}}\div(\frac{1}{4})^{-\frac{1}{2}}$.

22. $\dfrac{5}{3+3^{-2}}$.

23. $4^0+\left(\dfrac{5}{2}\right)^{-2}$.

24. $\dfrac{2^{-3}\cdot 3^{-2}\cdot 4^{-\frac{3}{2}}}{8^0\cdot 9^{-1}\cdot 8^{-\frac{5}{3}}}$.

25. Express .000000003 in a briefer form by the use of negative exponents. How many more figures and symbols are there in the first form than in the second?

26. Express .000000001 in a briefer form by the use of negative exponents.

27. Express 10^{-7} as a decimal fraction. Also 17×10^{-8}.

Give the value of

28. 5^0, 500^0, $\left(\dfrac{1}{5}\right)^0$, $\left(\dfrac{x}{y}\right)^0$, $(-7)^0$, $(a+b)^0$, $\dfrac{3^0}{2^0}$.

29. 8×5^0, $5\left(\dfrac{2}{3}\right)^0$, $3\,a^0$, $(3\,a)^0$, $9^{\frac{1}{2}}\div 3^0$, $\dfrac{7}{5^0}$.

30. $1^{\frac{6}{5}}$, $1^{-\frac{5}{3}}$, 1^0, $1^{\frac{2}{3}}\times 1^{-\frac{5}{4}}$, $9^{\frac{3}{2}}\times 1^{\frac{4}{3}}$, $5-4^0\times 1^{-\frac{2}{5}}$.

31. $8^{-\frac{2}{3}}\times 16^{\frac{3}{4}}\times 2^0$. **32.** $4^{-2}\times 8^{\frac{2}{3}}+2^{-3}$.

33. Express x^0 as some power of x divided by itself.

34. Express 4^{-2} as the quotient of two powers of 4.

35. State the value of \$$4^0$. Of \$$4^{-2}$. \$$4^{-\frac{3}{2}}$.

36. Which is the greater, $(\frac{1}{4})^{-2}$ or $(\frac{1}{4})^{-3}$?

Simplify the following by performing the indicated operations, and reducing the results:

37. $6\,a^{\frac{1}{2}}x^{-\frac{2}{3}}\cdot a^{\frac{1}{2}}x^{\frac{1}{3}}$.

38. $8\,x^{-\frac{2}{3}}y \div 4\,x^{\frac{1}{3}}y^2$.

39. $8\,a^{-2} \div 2\,a^{-3}$.

40. $x^2\sqrt[3]{-8\,x^{-2}}$.

41. $\dfrac{a\sqrt{a^{-2}}}{3\sqrt[3]{a^{-1}}}$.

42. $\dfrac{x^{2a+b}\cdot x^{2a-b}}{x^{a+b}\cdot x^{a-b}}$.

43. $\dfrac{7\,b^5\sqrt[3]{a^{-5}}}{3\,b^{-5}\sqrt[3]{a^{-4}}}$.

44. $a^n y^{-\frac{m}{2}} \div y^m\sqrt{a^{-2n}}$.

45. $\dfrac{a^{-\frac{2}{n}}\sqrt{b^n}}{b^{-\frac{n}{2}}\sqrt{a}}$.

46. $\dfrac{p^{-\frac{2}{n}}\sqrt[2n]{p^3}}{p\sqrt[n]{p^{-3}}}$.

Give the value of

47. $4^{-\frac{3}{2}}+8^{\frac{1}{3}}-(\frac{1}{8})^{-\frac{2}{3}}$.

48. $4^0+3x^0-5\times 4^{-\frac{3}{2}}$.

49. $\dfrac{2^{-2}}{3^{-2}-5^{-2}}$.

50. $\dfrac{1}{a^{-1}+b^{-1}}$.

51. $\dfrac{x^{-2}}{x^{-1}-y^{-1}}$.

62. Meaning of $(a^m)^n$ for Fractional and Negative Exponents. We now extend the law $(a^m)^n=a^{mn}$ to fractional and negative exponents.

Ex. 1. Find the value of $(9^{\frac{5}{2}})^{-\frac{6}{5}}$.

$$(9^{\frac{5}{2}})^{-\frac{6}{5}}=9^{-3}=\frac{1}{9^3}=\frac{1}{729}. \quad \textit{Ans.}$$

Ex. 2. Find the value of $(9\,a^{-2}x^{\frac{1}{2}}\,y^{-3})^{-\frac{3}{2}}$.

$$(9\,a^{-2}x^{\frac{1}{2}}y^{-3})^{-\frac{3}{2}}=9^{-\frac{3}{2}}a^3x^{-\frac{3}{4}}y^{\frac{9}{2}}$$

$$=\frac{a^3y^{\frac{9}{2}}}{27\,x^{\frac{3}{4}}}. \quad \textit{Ans.}$$

Ex. 3. $\sqrt{\sqrt{x}}=\sqrt{x^{\frac{1}{2}}}=(x^{\frac{1}{2}})^{\frac{1}{2}}=x^{\frac{1}{4}}$. *Ans.*

Ex. 4. $(\sqrt{ax})^4=(a^{\frac{1}{2}}x^{\frac{1}{2}})^4=a^2x^2$. *Ans.*

Hence, in general, to simplify a complex expression in exponents,

Convert each radical sign into a fractional exponent;

Convert each power of a power into a power with a single exponent;

Convert each negative exponent into a positive exponent;

Simplify by cancellations and collections.

EXERCISE 57

Reduce to simplest form:

1. $(4^3)^{-\frac{2}{3}}$.

2. $(x^{-3})^{\frac{2}{3}}$.

3. $(5^4)^0$.

4. $(64^{\frac{1}{3}})^{\frac{3}{2}}$.

5. $(4\,a^{\frac{1}{3}})^{\frac{3}{2}}$.

6. $(9\,b^{-\frac{1}{2}})^{-2}$.

7. $2^{a+b}\cdot 2^{a-b}$.

8. $(2^{a+b})^{a-b}$.

9. $(x^{-\frac{1}{3}}y^{\frac{1}{2}})^{-6}$.

10. $(4\,a^{-2}b^{-3}x)^{\frac{1}{2}}$.

11. $(27^{\frac{5}{3}})^{-\frac{3}{5}}$.

12. $(5\,a^{-4})^0$.

13. $(-3\,a^2b^{-\frac{1}{3}})^{-3}$.

14. $(x^{p+q}\cdot x^{p-q})^{\frac{1}{2}}$.

15. $(49\,x^{-\frac{2}{3}}y^4)^{-\frac{1}{2}}$.

16. $\sqrt{\sqrt[3]{a}}$.

17. $\sqrt{\sqrt{a}}$.

18. $\sqrt[4]{\sqrt[3]{x}}$.

19. $\sqrt{\sqrt{5}}$.

20. $(\sqrt[4]{x})^2$.

21. $(\sqrt{ax})^6$.

22. $(\sqrt[3]{ax})^6$.

23. $(\sqrt[3]{xy^4})^{12}$.

24. $[\sqrt{(a-b)^3}]^4$.

25. $\sqrt[3]{x^{-1}\sqrt{y^3}}$.

26. $\left(\dfrac{25\sqrt{x}}{9\sqrt[3]{x^{-2}}}\right)^{-\frac{3}{2}}$.

27. $\dfrac{(a^n)^{n+2}}{a^{n+1}\cdot a^{n-1}}$.

28. $\left(\dfrac{27\,x\sqrt[3]{y^{-2}}}{8\,y^{\frac{1}{3}}\sqrt{x^{-2}}}\right)^{-\frac{3}{2}}$.

63. Polynomials whose Terms Contain Fractional or Negative Exponents.

Ex. 1. Multiply

$$x^{-1}-2\,x^{-\frac{1}{2}}y^{\frac{1}{2}}+4\,y \text{ by } x^{-1}+2\,x^{-\frac{1}{2}}y^{\frac{1}{2}}+4y.$$

$$\begin{array}{lllll}
x^{-1}-2\,x^{-\frac{1}{2}}y^{\frac{1}{2}}+4\,y & & & & \\
x^{-1}+2\,x^{-\frac{1}{2}}y^{\frac{1}{2}}+4\,y & & & & \\
\hline
x^{-2}-2\,x^{-\frac{3}{2}}y^{\frac{1}{2}} & +4\,x^{-1}y & & & \\
\quad +2\,x^{-\frac{3}{2}}y^{\frac{1}{2}} & -4\,x^{-1}y & +8\,x^{-\frac{1}{2}}y^{\frac{3}{2}} & & \\
 & +4\,x^{-1}y & -8\,x^{-\frac{1}{2}}y^{\frac{3}{2}} & +16\,y^2 & \\
\hline
x^{-2} & +4\,x^{-1}y & & +16\,y^2. & \textit{Product.}
\end{array}$$

Ex. 2. Extract the square root of

$$a^{-\frac{3}{2}}+\frac{8}{a}-2a^{-\frac{3}{4}}+\frac{16}{\sqrt{a}}-8a^{-\frac{1}{4}}+1.$$

Writing the expression by use of exponents only,

$$\begin{array}{l|l|l}
 & a^{-\frac{3}{2}}+8a^{-1}-2a^{-\frac{3}{4}}+16a^{-\frac{1}{2}}-8a^{-\frac{1}{4}}+1 & \underline{a^{-\frac{3}{4}}+4a^{-\frac{1}{4}}-1} \\
 & a^{-\frac{3}{2}} & \\
\hline
2a^{-\frac{3}{4}}+4a^{-\frac{1}{4}} & 8a^{-1}-2a^{-\frac{3}{4}}+16a^{-\frac{1}{2}} & \\
 & 8a^{-1}\qquad\qquad +16a^{-\frac{1}{2}} & \\
\hline
2a^{-\frac{3}{4}}+8a^{-\frac{1}{4}}-1 & -2a^{-\frac{3}{4}}-8a^{-\frac{1}{4}}+1 & \\
 & -2a^{-\frac{3}{4}}-8a^{-\frac{1}{4}}+1 & \\
\hline
\end{array}$$

EXERCISE 58

1. Arrange $5x^{-2}+3+4x^2-3x-x^{-1}$ in descending order of magnitude.

Also arrange $a^{-\frac{n}{2}}-a^{-n}+a^n-1+a^{\frac{n}{2}}$ in ascending order.

Multiply:

2. $3x^{\frac{3}{4}}-2x^{\frac{1}{2}}+3x^{\frac{1}{4}}+4$ by $2x^{\frac{1}{4}}-3$.
3. $3a^{-2}-3a^{-1}+2$ by $3a^{-2}-2a^{-1}$.
4. $3a^{\frac{4}{3}}-2a^{\frac{2}{3}}b^{\frac{2}{3}}-4b^{\frac{4}{3}}$ by $4a^{\frac{2}{3}}-3b^{\frac{2}{3}}$.
5. $3x^{\frac{2}{3}}-5x^{\frac{1}{3}}+4$ by $3-4x^{-\frac{1}{3}}$.
6. $x^{-2}-x^{-1}y+y^2$ by $x^{-2}+x^{-1}y+y^2$.
7. $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+2y^{\frac{2}{3}}$ by $y^{-\frac{2}{3}}+x^{-\frac{1}{3}}y^{-\frac{1}{3}}+2x^{-\frac{2}{3}}$.

Rearrange and then multiply:

8. $2x^2+5-3x$ by $2x^{-1}-6+3x^{-2}$.
9. $3x^{\frac{4}{3}}+2x^{-\frac{2}{3}}-x^{\frac{1}{3}}$ by $x^{-\frac{1}{3}}+2x^{\frac{2}{3}}$.

Divide:

10. $4x^{-2}-4x^{-1}+1$ by $2x^{-1}-1$.

11. $6\,a^{\frac{4}{3}}-a+1$ by $2\,a^{\frac{2}{3}}+a^{\frac{1}{3}}$.

12. $3\,a^{-4}-4\,a^{-2}-4$ by $3\,a^{-2}+2$.

13. $x^{-3}-y^{3}$ by $x^{-1}-y$.

14. $a^{-2}+a^{-1}b+b^{2}$ by $a^{-1}-a^{-\frac{1}{2}}b^{\frac{1}{2}}+b$.

15. $2\,a^{-1}+3+a+6\,a^{2}$ by $2a^{-\frac{2}{3}}-a^{\frac{1}{3}}+3\,a^{\frac{4}{3}}$.

Extract the square root of

16. $a^{-4}-6\,a^{-3}+13\,a^{-2}-12\,a^{-1}+4$.

17. $4\,x^{2}-12\,x^{\frac{3}{2}}+7\,x+3\,x^{\frac{1}{2}}+\frac{1}{4}$.

18. $9\,x^{\frac{4}{3}}-12\,x-26\,x^{\frac{2}{3}}+20\,x^{\frac{1}{3}}+25$.

19. $4\,a^{4}+12a^{3}b^{-1}+a^{2}b^{-2}-12\,ab^{-3}+4\,b^{4}$.

20. $16\,x^{\frac{2}{3}}+24\,a^{\frac{1}{2}}x^{\frac{1}{3}}-7\,a-12\,a^{\frac{3}{2}}x^{-\frac{1}{3}}+4\,a^{2}x^{-\frac{2}{3}}$.

21. $13-12\,b^{-\frac{2}{3}}+b^{\frac{4}{3}}-6\,b^{\frac{2}{3}}+4\,b^{-\frac{4}{3}}$.

EXERCISE 59

ORAL

Give the value of each of the following:

1. $4^{\frac{3}{2}},\ 4^{-\frac{3}{2}},\ 4^{-2},\ 4^{0},\ (4^{-2})^{-\frac{1}{2}},\ (4^{-2})^{\frac{3}{2}},\ \frac{1}{4^{-2}}$.

2. $(\frac{4}{9})^{\frac{3}{2}},\ (\frac{4}{9})^{0},\ (\frac{4}{9})^{-\frac{1}{2}},\ 4^{\frac{1}{2}}+(\frac{4}{9})^{\frac{1}{2}}$.

3. $x^{a+b}\cdot x^{2a-b},\ (x^{a+b})^{a-b}$.

4. $x^{2a+b}x^{2a-b}(x^{2a+b})^{2a-b}$.

5. $x^{\frac{n+1}{2}}\cdot x^{\frac{n-1}{2}}$.

6. $2^{n+1}\cdot 2^{n-1}$.

7. $2^{\frac{1}{2}}\cdot 2^{\frac{3}{2}},\ 3^{\frac{3}{2}}\cdot 3^{-\frac{1}{2}}$.

8. $4^{\frac{1}{2}}\cdot 4^{2},\ 4^{\frac{3}{2}}\cdot 4^{0},\ 7\,x^{0}+5\,a^{0}$.

9. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})$.

10. $(a^{\frac{3}{2}}+b^{\frac{3}{2}})(a^{\frac{3}{2}}-b^{\frac{3}{2}})$.

11. $(a^{\frac{3}{2}}-b^{\frac{3}{2}})\div(a^{\frac{1}{2}}-b^{\frac{1}{2}})$.

12. $(x^{-n}+y)(x^{-n}-y)$.

13. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^{2}$.

14. $(a^{-\frac{3}{2}}+b^{\frac{1}{2}})^{2}$.

15. $\dfrac{x^{-3}-y^{-3}}{x^{-1}-y^{-1}}$.

Factor:

16. $a^{\frac{2}{3}}-b^{\frac{2}{3}}$.

17. $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.

18. $a^{-2}-b^{-2}$.

19. $x^{-4}-9$.

20. Give the value of each of the following: $\frac{3^0}{5}$, $\frac{3}{5^0}$, $\frac{3^0}{5^0}$, $3^0\times 5$, 3×5^0, $3^0\times 5^0$, 3^0+5^0, 3^0-5, $\frac{1}{2^{-1}+3}$, $\frac{1}{a^{-1}+b^{-1}}$.

21. Give the square root of each of the following: $4x^{-\frac{2}{3}}$, $4x^{-\frac{2}{5}}$, $4x^{-3}$, $a^{-3}b^{\frac{1}{2}}$, $ax^{6}y^{-\frac{1}{2}}$, $\frac{1}{4}a^{3}b^{\frac{1}{2}}$.

22. Give the value of each of the following: $(.3)^{-2}$, $1^{-2}\times 1^0\times 1^{-\frac{2}{3}}$, $(.01)^{\frac{1}{2}}$, $1^{-\frac{2}{3}}\div 4^0$, $(.25)^{\frac{1}{2}}$, $(.25)^{\frac{3}{2}}$.

23. Give the reciprocal of 2. Of $\frac{2}{3}$, $-\frac{3}{4}$, 4^{-2}, $8^{-\frac{2}{3}}$, $1^{-\frac{2}{3}}$.

EXERCISE 60

REVIEW

Simplify:

1. $\left(\frac{4}{9}\right)^{\frac{3}{2}}+4^0-8^{-\frac{2}{3}}$. Also $\left(\frac{4}{9}\right)^{\frac{3}{2}}\cdot 4^0\cdot 8^{-\frac{2}{3}}$.

2. $\frac{7a^5\sqrt[3]{x^{-5}}}{3a^{-5}\sqrt[3]{x^{-4}}}$.

3. $\left(\frac{16x^{-4}}{81y^3}\right)^{-\frac{3}{4}}$.

4. $\frac{(x^{n-1})^{n}}{x^n\cdot x^{n-1}}$.

5. $(64^{-1})^{-\frac{2}{3}}$.

6. $(\sqrt{a^3x^3})^4$.

7. $\sqrt[4]{\sqrt{x^5}}$.

8. $(9a^{-2}x^{\frac{1}{2}}y^{-3})^{-\frac{3}{2}}$.

9. Multiply $5x^{\frac{2}{3}}-3x^{\frac{1}{3}}+7$ by $2-3x^{-\frac{1}{3}}+2x^{-\frac{2}{3}}$.

10. Divide $x^{-\frac{3}{2}}-y^3$ by $x^{-\frac{1}{2}}-y$.

11. Arrange and extract the square root of

$$x^{-2}+y^{\frac{4}{3}}+x^{-\frac{2}{3}}y^{\frac{1}{3}}-2x^{-\frac{1}{2}}y-x^{-1}y^{\frac{2}{3}}.$$

12. Express $19^{-\frac{2}{3}}$ in two other forms.

13. Divide $\frac{1}{x^{-\frac{3}{5}}}-\frac{1}{y^{-\frac{3}{5}}}$ by $\sqrt[5]{x}-\sqrt[5]{y}$.

14. Simplify the following and express the result with positive exponents:

$$\sqrt{(a^{-1}c^{-1}p^2qr^{-3})(a^{-1}b^3p^{-3}q^2r)\div(bc^{-3}p^{-1}q^2r^{-2})}.$$

15. Evaluate $\dfrac{1.4\times10^{-3}\times2.7\times10^2}{4.2\times10^{-7}\times8.1\times10^3}$.

16. Simplify $6\,a^{-4}b^7\div3\sqrt{a^{-2}b^8}$.

17. Simplify $10^{3.17625}\div10^{1.29164}$.

18. Express one millionth of one millionth of an inch in a short way by use of a negative exponent.

19. Express 213×10^{-10} as a decimal fraction.

20. Multiply $6+5\,a^{-2}-3\,a^{-1}$ by $3\,a^2+2\,a-3$, first arranging both expressions in ascending powers of a.

21. Evaluate $(.125)^{-\frac{2}{3}}+\dfrac{3}{2+2^{-1}}$.

22. Evaluate $.6\times32^0$; $.8\times4^{-2}$; $12\times9^{-\frac{1}{2}}$.

23. Which is greater $(2^3)^{-4}$ or $2^3\times2^{-4}$? How many times greater?

24. Does $(x^a)^2$ equal x^{a^2}? Illustrate numerically.

25. Divide $a-b^2$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

64. Indicated Roots. The root of a quantity may be indicated in two different ways:

(1) By the use of a fractional exponent; as $a^{\frac{1}{3}}$.
(2) By the use of a radical sign; as $\sqrt[3]{a}$.

For some purposes, one of these methods is better; for some, the other method.

Thus, when we have $b^{\frac{2}{3}}\times b^{\frac{1}{3}}\div b^{-\frac{1}{2}}$, where the quantities are alike except in their exponents, it is usually better to use fractional exponents to indicate roots. But if we have $7\sqrt{12}-6\sqrt{3}+2\sqrt{48}$, where exponents are alike, but coefficients and bases unlike, it is usually an advantage to use the radical sign to indicate roots.

65. A Radical is a root of a quantity indicated by the use of the radical sign; as $\sqrt{a}$, $\sqrt[3]{8}$.

The **radicand** is the quantity under the radical sign.

In treating radicals, we deal only with principal roots (see § 51, p. 102), unless the contrary is stated.

66. Surds. An indicated root which may be exactly extracted is said to be *rational;* as $\sqrt[3]{27}$, since the cube root of 27 is 3.

A **surd** is an indicated root which cannot be exactly extracted; as $\sqrt{5}$, $\sqrt[3]{4}$.

67. The Coefficient of a radical is the number prefixed to the radical proper, to show how many times the radical is taken.

Thus, the coefficient of $3\sqrt{7}$ is 3; of $5(a+b)\sqrt{x}$ is $5(a+b)$.

68. Entire Surds. If a surd has unity for its coefficient, it is said to be *entire.*

69. The Degree of a radical is the number of the indicated root.

Thus, $\sqrt[3]{5}$ is a radical of the third degree.

70. Similar Radicals are those which have the same quantity under the radical sign and the same index. (The coefficients and signs of the radicals may be unlike. Hence, similar radicals must be alike in two respects, and may be unlike in two other respects.)

Thus, $3\sqrt{7}$ and $-5\sqrt{7}$ are similar radicals.

71. Fundamental Principles. Since a radical and a quantity affected by a fractional exponent differ only in form, in investigating the properties of radicals we may use the properties obtained for fractional exponents.

Thus, since $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$.

$$\therefore \sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}, \quad \text{or} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

Similarly, from the properties of exponents,

$$\sqrt{a} \div \sqrt{b} = \sqrt[n]{\frac{a}{b}}.$$

and $(\sqrt{a})^n = (a^{\frac{1}{m}})^n = a^{\frac{n}{m}} = \sqrt{a^n}$.

72. Simplification of a Quantity under the Radical Sign.

Ex. 1. Simplify $\frac{1}{10}\sqrt[3]{250\,a^2x^6}$.

$$\tfrac{1}{10}\sqrt[3]{250\,a^2x^6} = \tfrac{1}{10}\sqrt[3]{125\,x^6 \times 2\,a^2} = \tfrac{1}{2}\,x^2\sqrt[3]{2\,a^2}. \quad \textit{Ans.}$$

Ex. 2. Simplify $\frac{2\,a}{x}\sqrt{\frac{8\,x^2}{27\,a}}$.

$$\frac{2\,a}{x}\sqrt{\frac{8\,x^2}{27\,a}} = \frac{2\,a}{x}\sqrt{\frac{8\,x^2}{27\,a} \times \frac{3\,a}{3\,a}} = \frac{2\,a}{x}\sqrt{\frac{24\,ax^2}{81\,a^2}}.$$

$$= \frac{2\,a}{x}\sqrt{\frac{4\,x^2}{81\,a^2} \times 6\,a} = \frac{4}{9}\sqrt{6\,a}. \quad \textit{Ans.}$$

EXERCISE 61

Express in the simplest form:

1. $\sqrt[3]{16\,a^4xy^2}$.
2. $\sqrt{16\,a^4xy^2}$.
3. $\sqrt[4]{16\,a^4xy^2}$.
4. $\sqrt{54\,a^3b^2c^4}$.
5. $\frac{2}{3}\sqrt{54a^3b^2c^4}$.
6. $-\frac{1}{4}\sqrt[3]{25a^3b}$.
7. $\sqrt[4]{80\,x^6y^{12}}$.
8. $ax\sqrt{a^2x^4y}$.
9. $\sqrt[5]{64\,ax^{10}}$.
10. $\sqrt[6]{729\,b^3y^9}$.
11. $\sqrt[3]{432\,a^2x^{3n}}$.
12. $\sqrt[n]{a^{n+2}x^{2n}}$.
13. $\sqrt{(4\,x^2+4\,x+1)y}$.
14. $\sqrt{(a+b)(a^2-b^2)}$.
15. $\sqrt{\frac{12\,a^2x^5}{25\,b^4}}$.
16. $\sqrt{72 \times 40}$.
17. $\sqrt{\frac{36(2\,a-b)^3}{49(x-y)^2}}$.

Simplify:

18. $10\sqrt{\frac{3}{5}}$. **19.** $\sqrt[3]{\frac{3}{4}}$. **20.** $\sqrt{3\frac{1}{7}}$. **21.** $\sqrt[4]{\frac{3}{8}}$.

22. $\frac{3}{2}\sqrt{\frac{20\,a^2}{x}}$. **24.** $\sqrt[3]{\frac{9\,a^2}{50}}$. **26.** $\sqrt[4]{\frac{3\,ax^6}{8\,y}}$.

23. $\sqrt{\frac{4}{25\,a}}$. **25.** $\sqrt[3]{\frac{8}{7\,a^2}}$. **27.** $\frac{1-x}{1+x}\sqrt{\frac{1+x}{1-x}}$.

28. $(a^2-b^2)\sqrt[3]{\frac{(a+b)^2}{(a-b)^5}}$. **30.** $\sqrt[n]{\frac{a^{3n+1}}{b^{2n-3}}}$.

29. $\sqrt[n]{\frac{a}{b^{n-1}}}$. **31.** $\sqrt[n+1]{\frac{(a+b)^{n+4}}{(a-b)^{n+3}}}$.

32. Given $\sqrt{3}=1.73205^+$, find the value of $\sqrt{108}$ to four decimal places in the shortest way. Also that of $\sqrt{192}$.

33. Using $\sqrt{6}=2.44948^+$, make up and work an example similar to Ex. 32.

73. Making Entire Surds. It is sometimes desirable to introduce the coefficient of a radical under the radical sign. This may be done by reversing the process of § 72.

Ex. 1. Express $3\sqrt[3]{5}$ as an entire surd.

$$3\sqrt[3]{5}=\sqrt[3]{3^3\times 5}=\sqrt[3]{135}.\quad \textit{Ans.}$$

Ex. 2. $-2\sqrt[5]{3}=-\sqrt[5]{96}=\sqrt[5]{-96}$. *Ans.*

EXERCISE 62

Express as entire surds:

1. $5\sqrt{3}$.

2. $5\sqrt[3]{3}$.

3. $5\sqrt[4]{3}$.

4. $\frac{1}{2}\sqrt{\frac{4}{3}}$.

5. $\frac{1}{2}\sqrt[3]{\frac{4}{3}}$.

6. $\frac{4}{5\,a}\sqrt{\frac{25\,a}{2}}$.

7. $\frac{3\,a}{2\,b}\sqrt{\frac{8\,b}{27\,a^2}}$.

8. $\frac{a}{b}\sqrt[3]{b^{n-3}}$.

9. $\frac{2}{9}\sqrt[3]{3\frac{3}{8}}$.

10. $\frac{a}{x^n}\sqrt{\frac{x^{3n}}{a^{n+2}}}$.

11. $\frac{1}{2}\sqrt[4]{\frac{4}{3}}$.

12. $\frac{3}{5}\sqrt{\frac{10}{3}}$.

13. $x^3\sqrt{1-\frac{a^2}{x^2}}$.

14. $3\,x^2\sqrt{\frac{a-b}{x^4}}$.

15. $x^3\left(1-\frac{a^2}{x^2}\right)^{\frac{3}{2}}$.

16. $(a-b)\sqrt{3-\frac{2\,b}{a-b}}$.

74. Simplification of Indices. If the exponent of the quantity under the radical sign and the index of the radical sign have a common factor, this factor may be canceled and the radical thereby simplified.

Ex. 1. $\sqrt[6]{a^2}=a^{\frac{2}{6}}=a^{\frac{1}{3}}=\sqrt[3]{a}$. *Ans.*

Ex. 2. $\sqrt[6]{125}=\sqrt[6]{5^3}=\sqrt{5}$. *Ans.*

EXERCISE 63

Simplify the indices of the following:

1. $\sqrt[4]{a^2}$.
2. $\sqrt[6]{a^3}$.
3. $\sqrt[4]{a^2y^4}$.
4. $\sqrt[4]{49}$.
5. $\sqrt[6]{27\,a^3}$.
6. $\sqrt[4]{100\,a^2x^8}$.
7. $\sqrt[9]{8\,a^3b^6c^9}$.
8. $\sqrt[12]{81\,a^4x^8}$.
9. $\sqrt[10]{32\,a^5x^{10}y^{15}}$.
10. $\sqrt[2m]{9\,x^2y^4z^{10}}$.
11. $\sqrt[5n]{x^ny^{2n}z^{3n}}$.
12. $6\sqrt[4]{2\frac{7}{9}}$.
13. $\sqrt[4]{a^2-4\,ab+4\,b^2}$.
14. $\sqrt[6]{8(a-2\,b)^3}$.

75. Addition of Radicals.

Ex. $6\sqrt{4\frac{1}{6}}-10\sqrt{3\frac{1}{5}}-5\sqrt{\frac{4}{5}}+7\sqrt{24}$

$$=6\sqrt{\tfrac{25}{6}}-10\sqrt{\tfrac{16}{5}}-5\sqrt{\tfrac{4}{5}}+14\sqrt{6}$$

$$=5\sqrt{6}-8\sqrt{5}-2\sqrt{5}+14\sqrt{6}$$

$$=19\sqrt{6}-10\sqrt{5}.\quad \textit{Ans.}$$

76. Multiplication and Involution of Radicals.

Ex. 1. $5\sqrt{6}\times 2\sqrt{3}=10\sqrt{18}.$

$$=30\sqrt{2}.\quad \textit{Ans.}$$

Ex. 2. $(\sqrt{3})^8=(3^{\frac{1}{2}})^8=3^4=81.$ *Ans.*

EXERCISE 64

Simplify and collect:

1. $2\sqrt{27}-3\sqrt{18}+\sqrt{300}-\sqrt{162}.$
2. $\frac{2}{3}\sqrt[3]{\frac{3}{4}}-\frac{5}{2}\sqrt[3]{\frac{2}{9}}+\sqrt[3]{48}.$
3. $a\sqrt[3]{2x}+\sqrt[3]{128a^3x}-\dfrac{3}{a}\sqrt[3]{250a^6x}.$
4. $\frac{2}{3}\sqrt{162x^2}+20x\sqrt{4\frac{1}{2}}-\dfrac{15}{x^2}\sqrt{2x^6}-x^2\sqrt{\dfrac{8}{x^2}}.$
5. $\sqrt{24a^2}-6a\sqrt{\dfrac{1}{6}}+\dfrac{2a}{5}\sqrt{\dfrac{25}{6}}-a\sqrt{66\frac{2}{3}}+\dfrac{1}{2a}\sqrt{96a^4}.$
6. $5\sqrt{(a-b)^2x}-3\sqrt{(2a-3b)^2x}+\sqrt{a^2x+2abx+b^2x}.$
7. $\sqrt{(a-b)^2y}+\sqrt{(a+b)^2y}-\sqrt{a^2y}+\sqrt{(1-a)^2y}-\sqrt{y}.$
8. $4\sqrt{\frac{1}{6}}-5\sqrt{6}-3\sqrt{\frac{2}{3}}+2\sqrt[4]{36}.$

9. Compute to three decimal places the numerical value of $\sqrt{50}+\sqrt{98}-\sqrt{72}$ without first simplifying and collecting the radicals. Then simplify first and compute. Compare the amount of work in the two processes.

10. State some of the advantages in being able to simplify radicals.

Multiply:

11. $\frac{2}{3}\sqrt{\frac{5}{2}}$ by $\frac{1}{2}\sqrt{\frac{4}{5}}.$ 12. $(3\sqrt{a}+5\sqrt{b})^2.$

13. $4\sqrt[3]{12}$ by $3\sqrt[3]{4}$.

14. $(\sqrt{3\,a}+\sqrt{5\,b})^2$.

15. $a\sqrt{a}-b\sqrt{b}$ by $3\sqrt{ab}$.

16. $\sqrt{5-\sqrt{7}}$ by $\sqrt{5+\sqrt{7}}$.

17. $\frac{2}{3}\sqrt{8}+2\sqrt{32}-\frac{1}{2}\sqrt{48}$ by $2\sqrt{8}-\frac{1}{4}\sqrt{32}+\sqrt{12}$.

18. $10\sqrt{\frac{1}{5}}-4\sqrt{\frac{1}{2}}+\frac{1}{2}\sqrt{500}$ by $\sqrt{\frac{4}{5}}+8\sqrt{2}-\frac{7}{5}\sqrt{5}$.

19. $\sqrt{x+2}-\sqrt{x-2}$ by $\sqrt{x+2}+\sqrt{x-2}$.

20. $\frac{1}{2}\sqrt{a^2-b^2}+\frac{1}{2}\sqrt{a^2+b^2}$ by $\frac{1}{2}\sqrt{a^2-b^2}-\frac{1}{2}\sqrt{a^2+b^2}$.

21. In the shortest way find the value of

$$(3\sqrt{2}-\sqrt{5})(4\sqrt{3}+\sqrt{7})(3\sqrt{2}+\sqrt{5})(4\sqrt{3}-\sqrt{7}).$$

22. $(a+\sqrt{5\,b})^2-(a-\sqrt{5\,b})^2$.

23. $(\sqrt{2})^8$; $(\sqrt{2})^{10}$; $(\sqrt{3})^6$; $(-\sqrt{3})^7$; $(\frac{1}{2}\sqrt{2})^6$.

77. Division of Radicals. Reversing the process for multiplication, we have the following rule for dividing one radical by another:

Find the quotient of the coefficients for a new coefficient, and the quotient of the quantities under the radical signs for a new quantity under the radical;

Simplify the result.

Ex. Divide $6\sqrt{8}$ by $3\sqrt{6}$.

$$\frac{6\sqrt{8}}{3\sqrt{6}}=2\sqrt{\tfrac{4}{3}}=2\sqrt{\tfrac{12}{9}}=\tfrac{4}{3}\sqrt{3}.\quad \textit{Quotient.}$$

EXERCISE 65

Divide:

1. $\sqrt{27}$ by $\sqrt{3}$.

2. $8\sqrt{125}$ by $10\sqrt{10}$.

3. $3\sqrt{405}$ by $9\sqrt{45}$.

4. $\sqrt{\frac{15}{16}}$ by $\sqrt{\frac{5}{8}}$.

5. $5\sqrt{\frac{22}{35}}$ by $2\sqrt{\frac{7}{11}}$.

6. $\sqrt{ab}$ by $\sqrt{bc}$.

7. $a^3\sqrt{ab^3}$ by $2a\sqrt{a^3b}$. **8.** $12\sqrt{7}-60\sqrt{5}$ by $4\sqrt{3}$.

9. $6\sqrt{105}+18\sqrt{40}-45\sqrt{12}$ by $3\sqrt{15}$.

78. Rationalizing a Denominator.

Ex. 1. $\dfrac{5}{4\sqrt[3]{a}}=\dfrac{5}{4\sqrt[3]{a}}\times\dfrac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}}=\dfrac{5\sqrt[3]{a^2}}{4a}$. *Ans.*

Ex. 2. $\dfrac{5}{3\sqrt{a}-2\sqrt{b}}=\dfrac{5}{3\sqrt{a}-2\sqrt{b}}\times\dfrac{3\sqrt{a}+2\sqrt{b}}{3\sqrt{a}+2\sqrt{b}}$

$=\dfrac{15\sqrt{a}+10\sqrt{b}}{9a-4b}$. *Ans.*

EXERCISE 66

Reduce to an equivalent fraction with rational denominator:

1. $\dfrac{4}{3\sqrt{7}}$.

2. $\dfrac{\sqrt{7}}{\sqrt{3}}$.

3. $\dfrac{5}{3\sqrt{2x}}$.

4. $\dfrac{6}{\sqrt[3]{4a}}$.

5. $\dfrac{8}{\sqrt[3]{2}}$.

6. $\dfrac{5}{\sqrt[3]{a+b}}$.

7. $\dfrac{8}{3\sqrt[3]{2y}}$.

8. $\dfrac{\sqrt{a+b}}{\sqrt{a-b}}$.

9. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a-b}}$.

10. $\dfrac{5-7\sqrt[3]{5}}{4\sqrt[3]{3}}$.

11. $\dfrac{4\sqrt{5}-2\sqrt{2}}{3\sqrt{5}+4\sqrt{2}}$.

12. $\dfrac{3-\sqrt{x+2}}{3+\sqrt{x+2}}$.

13. $\dfrac{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}$.

14. $\dfrac{\sqrt{a+\sqrt{a^2-b^2}}}{\sqrt{a-\sqrt{a^2-b^2}}}$.

15. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}-\dfrac{3-\sqrt{5}}{3+\sqrt{5}}$.

16. $\dfrac{a}{\sqrt[n]{b}}$.

17. $\dfrac{a}{\sqrt[n-2]{b}}$.

18. Compute to the nearest thousandth the value of $\frac{3\sqrt{7}}{5\sqrt{5}}$ without rationalizing the denominator. Then rationalize the denominator and compute the numerical value of the result to the nearest thousandth. Compare the amount of work in the two processes.

Use the process of rationalizing the denominator as an aid in finding to the nearest thousandth the numerical value of

19. $\frac{2}{\sqrt{5}}$.

20. $\frac{5}{2\sqrt{7}}$.

21. $\frac{8}{\sqrt{200}}$.

22. $\frac{2\sqrt{7}}{3\sqrt{5}}$.

23. $\frac{3-\sqrt{5}}{2+\sqrt{5}}$.

24. $\frac{3\sqrt{6}-2\sqrt{5}}{2\sqrt{6}+\sqrt{5}}$.

25. In the formula $d=s\sqrt{2}$, in the shortest way find s to the nearest hundredth when $d=7$.

79. A Quadratic Surd is a surd of the second degree; as $\sqrt{3}$ and $\sqrt{ab}$.

A **binomial surd** is a binomial expression, at least one term of which contains a surd; as $\sqrt{2}+5\sqrt{3}$, or $a+\sqrt{b}$.

80. Finding the Square Root of a Binomial Surd by Inspection.

By actual multiplication we may find

$$(\sqrt{2}+\sqrt{5})^2=2+2\sqrt{10}+5=7+2\sqrt{10}.$$

In the square, $7+2\sqrt{10}$, 7 is the sum of 2 and 5, and 10 is the product of 2 and 5. Hence, in extracting the square root of a binomial surd,

Transform the surd term so that its coefficient shall be **2**;

Find two numbers such that their sum shall equal the rational term, and their product equal the quantity under the radical sign;

Extract the square root of each of these numbers, and connect the results by the proper sign.

Ex. Find the square root of $18+8\sqrt{5}$.

$$18+8\sqrt{5}=18+2\sqrt{80}.$$

The two numbers whose sum is 18 and product is 80 are 8 and 10.

$$\therefore \sqrt{18+8\sqrt{5}}=\sqrt{8}+\sqrt{10}.$$

$$=2\sqrt{2}+\sqrt{10}. \quad \textit{Root.}$$

EXERCISE 67

Find by inspection the square root of

1. $3+2\sqrt{2}$.
2. $9-2\sqrt{14}$.
3. $21+12\sqrt{3}$.
4. $17-12\sqrt{2}$.
5. $35-12\sqrt{6}$.
6. $9-6\sqrt{2}$.
7. $23-6\sqrt{10}$.
8. $18-12\sqrt{2}$.
9. $7+4\sqrt{3}$.
10. $26+4\sqrt{30}$.
11. $14+3\sqrt{3}$.
12. $2m+2\sqrt{m^2-n^2}$.

81. Equations Containing Radicals.

Ex. 1. Solve $\sqrt{x^2+7}-1=x$.

Transpose terms so that the radical shall be alone on one side of the equation.

$$\sqrt{x^2+7}=x+1.$$

Squaring, $$x^2+7=x^2+2x+1.$$

$$\therefore 2x=6.$$

$$x=3. \quad \textit{Root.}$$

CHECK: $$\sqrt{x^2+7}=\sqrt{9+7}=4.$$

$$x+1=3+1=4.$$

Observe that only the principal value of a radical is used in checking a result, as in the other processes in this chapter.

Ex. 2. Solve $\sqrt{x+3}+\sqrt{x}=5$.

Transpose terms so that one radical shall be alone on one side of the equation.

$$\sqrt{x+3}=5-\sqrt{x}.$$

Squaring, $$x+3=25-10\sqrt{x}+x.$$

$$\therefore 10\sqrt{x}=22.$$

$$5\sqrt{x}=11.$$

Squaring, $$25x=121.$$

$$x=\tfrac{121}{25}. \quad \textit{Root.}$$

Let the pupil check the work.

In general,

Transpose the terms of the given equation so that a single radical shall form one member of the equation.

Raise both members of the equation to the power indicated by the index of this radical;

Repeat the process if necessary.

Always check the root.

82. Fractional Equations Containing Radicals.

Ex. 1. $\sqrt{x}-\sqrt{x-8}=\dfrac{2}{\sqrt{x-8}}$.

Multiply by $\sqrt{x-8}$, $\sqrt{x^2-8x}-x+8=2$.

$$\therefore \sqrt{x^2-8x}=x-6.$$

$$x^2-8x=x^2-12x+36.$$

$$4x=36.$$

$$x=9. \quad \textit{Root.}$$

Ex. 2. $\dfrac{\sqrt{x}+3}{\sqrt{x}-2}=\dfrac{3\sqrt{x}-5}{3\sqrt{x}-13}$.

Clearing of fractions, $3x-4\sqrt{x}-39=3x-11\sqrt{x}+10.$

$$7\sqrt{x}=49.$$

$$\sqrt{x}=7.$$

$$x=49. \quad \textit{Root.}$$

EXERCISE 68

Solve the following equations:

1. $3-\sqrt{x+1}=0.$
2. $5-\sqrt{2x}=3.$
3. $1=\sqrt[3]{x}-1.$
4. $x-1-\sqrt{x^2+3}=0.$
5. $2\sqrt{3x-5}-3\sqrt{x+1}=0.$
6. $3\sqrt{x}-1=\sqrt{x}+1.$
7. $\sqrt{x+16}-8+\sqrt{x}=0.$
8. $\frac{3}{2}+\sqrt{x}=\sqrt{\frac{15}{4}+x}.$
9. $\sqrt{4x+3}=2\sqrt{x-1}+1.$
10. $2\sqrt{x}-\sqrt{4x-22}=\sqrt{2}.$
11. $\sqrt{25x-29}-\sqrt{4x-11}-3\sqrt{x}=0.$
12. $\sqrt{x}+\sqrt{4a+x}-2\sqrt{b+x}=0.$
13. $\sqrt{3+x}+\sqrt{x}=\dfrac{6}{\sqrt{3+x}}.$
14. $3\sqrt{2x+1}-3\sqrt{2x-3}=\dfrac{4}{\sqrt{2x-3}}.$
15. $\dfrac{\sqrt{x}-3}{\sqrt{x}+3}=\dfrac{\sqrt{x}+1}{\sqrt{x}-2}.$
16. $\dfrac{\sqrt{9x+2}-3\sqrt{x}}{\sqrt{9x+2}+3\sqrt{x}}=4.$
17. $\dfrac{6\sqrt{x}-7}{\sqrt{x}-1}-5=\dfrac{7\sqrt{x}-26}{7\sqrt{x}-21}.$
18. $\dfrac{1}{x}+\dfrac{1}{5}=\sqrt{\dfrac{1}{25}+\sqrt{\dfrac{1}{a^2x^2}+\dfrac{1}{x^4}}}.$

83. Extraneous Roots in Radical Equations. It may readily be shown that *squaring each of the two members of an equation does not necessarily produce an equivalent equation.*

Ex. $$5+\sqrt{x}=2.$$

Hence, $$\sqrt{x}=-3.$$

$$x=9.$$

Checking, $5+\sqrt{9}=2$, or $8=2$. But this is impossible; hence, 9 is not a root of the given equation.

Note that if the sign of the radical $\sqrt{x}$ is changed in the original equation, by solving the equation thus formed the result $x=9$ is obtained; this answer can be proved.

EXERCISE 69

Solve the following equations and check each result. In each case where the root is impossible, change the original equation so as to make the result obtained a root.

1. $1-\sqrt{x}=3.$ 2. $4-\sqrt{x+1}=5.$ 3. $3+\sqrt{2x}=5.$

4. $\sqrt{4x+9}+\sqrt{x}=\sqrt{x+5}.$ 5. $\sqrt{x+7}=2-\sqrt{x-5}.$

6. $\sqrt{x}-\sqrt{x+8}=8.$ 7. $\sqrt{x+8}-8+\sqrt{x}=0.$

8. $\sqrt{x+9}-\sqrt{x+2}-\sqrt{4x-27}=0.$

EXERCISE 70

ORAL REVIEW

Simplify at sight:

1. $\sqrt{12}.$
2. $\sqrt{x^5}.$
3. $2\sqrt{\frac{1}{2}}.$
4. $6\sqrt{\frac{1}{3}}.$
5. $4\sqrt{\frac{1}{5}}.$
6. $10\sqrt{\frac{2}{5}}.$
7. $x^2\sqrt{\frac{1}{x}}.$
8. $\sqrt{\frac{4}{3}}.$

9. $\sqrt[3]{8}$.

10. $\sqrt[3]{\frac{1}{4}}$.

11. $5\sqrt{\frac{2}{5}}$.

12. $\sqrt{\frac{R^2}{3}}$.

13. $\sqrt{\frac{2R^2}{3}}$.

14. $\frac{1}{3}\sqrt{\frac{3}{R^2}}$.

15. $\frac{1}{2}\sqrt{\frac{4R^2}{3}}$.

16. $6\sqrt{4\frac{1}{2}}$.

17. $\sqrt{\frac{1}{3}}$, $\frac{1}{\sqrt{3}}$, $\sqrt{8}$, $\sqrt{\frac{2}{3}}$, $\frac{2}{\sqrt{3}}$, $\frac{1}{\sqrt{a}}$, $\sqrt{\frac{1}{a}}$, $\sqrt{\frac{a}{b}}$, $\sqrt[4]{\frac{9}{16}}$.

18. $\frac{1}{3}+\sqrt[4]{9}$, $2\sqrt{3}+\sqrt[6]{27}$, $\sqrt[6]{4}+\sqrt[4]{25}$, $2-\sqrt[4]{4}$.

19. $\sqrt{18}+\sqrt{6}$, $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$, $\sqrt{4+\sqrt{5}}\times\sqrt{4-\sqrt{5}}$.

20. $(\sqrt{2})^6$, $(\sqrt{2})^5$, $(\sqrt{2})^7$, $(\sqrt{2})^{10}$, $(\sqrt{3})^{10}$.

21. In what respect is $\frac{\sqrt{15}}{3}$ a simpler expression than $\frac{\sqrt{5}}{\sqrt{3}}$?

At sight give the value of x in

22. $\sqrt{x}+1=4$.

23. $\sqrt{x+a}=b$.

24. $a+\sqrt{x}=c$.

25. $\sqrt{\frac{a}{x}}=b$.

26. $\sqrt{5x}=a$.

27. $\sqrt{ax}=c$.

EXERCISE 71

REVIEW

1. Simplify $4\sqrt{147}-3\sqrt{75}-6\sqrt{\frac{1}{3}}+18\sqrt{\frac{1}{27}}-24\sqrt{\frac{3}{4}}$.

2. Multiply $2+\sqrt{3}-\sqrt{5}$ by $2+\sqrt{3}+\sqrt{5}$.

3. Divide $6\sqrt{12}+3\sqrt{8}-6\sqrt{30}+4\sqrt{15}$ by $2\sqrt{6}$.

4. Rationalize the denominator of $\frac{2\sqrt{3}-3\sqrt{2}}{3-2\sqrt{6}}$.

Given $\sqrt{2}=1.41421^+$, $\sqrt{3}=1.73205^+$, compute in the shortest way to three decimal places the value of

5. $\frac{5}{\sqrt{3}}$.

6. $3\sqrt{15}\cdot\sqrt{30}$.

7. $5\sqrt{12}-2\sqrt{8}$.

8. Solve $2+\sqrt{x+3}=\sqrt{x-2}+3$.

9. Solve $\frac{4}{\sqrt{x-4}}-\sqrt{x}-\sqrt{x-4}=0$.

10. Simplify $2\sqrt[4]{9}+\sqrt{50}-6\sqrt{\frac{1}{2}}+\sqrt[3]{24}-2\sqrt[6]{27}-2\sqrt[4]{64}$.

11. Simplify $3\sqrt{5a^2}-10a\sqrt{\frac{1}{5}}-\frac{5}{a}\sqrt{\frac{49a^4}{5}}+2\sqrt{45a^2}$.

12. Simplify $(a+b)^2\sqrt{\frac{a-b}{a+b}}+(a-b)^2\sqrt{a^2-b^2}+\sqrt{(a^2-b^2)^3}$.

13. Rationalize the denominator of $\frac{4a}{3(2a)^{\frac{1}{3}}}$.

14. Multiply $\sqrt{a}+(a+x)^{\frac{1}{2}}-\sqrt{x}$ by $\sqrt{a}-\sqrt{a+x}-\sqrt{x}$.

15. Simplify $6\div\sqrt{3}+\sqrt{2}$. Also $6\div(\sqrt{3}+\sqrt{2})$.

16. Simplify $\frac{4\sqrt{7}}{\sqrt{7}-\sqrt{5}}$ and find its correct value to two decimal places.

17. Write into a single term $6\sqrt{2\frac{2}{3}a^3}-2(24ab^2)^{\frac{1}{2}}+a\sqrt{54a}$.

18. Simplify $\frac{2a}{3b}\left(\frac{45b^2x}{16a^4}\right)^{\frac{1}{2}}-\frac{10b}{a^2}\sqrt{\frac{a^2x}{5b^2}}+\frac{14x}{a^2b}\sqrt{\frac{5a^2b^2}{49x}}-\frac{5x}{2a}\left(\frac{1}{20x}\right)^{\frac{1}{2}}$.

84. An Imaginary Quantity is an indicated even root of a negative quantity; as $\sqrt{-4}$, $\sqrt[4]{-3}$, and $\sqrt{-a}$.

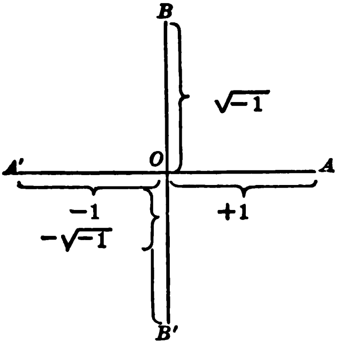
The term "imaginary" is used because, so long as we confine ourselves to plus quantity and to its direct opposite, minus quantity, there is no number which multiplied by itself will give a negative number, as -4, for instance. All the quantity considered hitherto, that is, all positive or negative quantity, whether it is rational or irrational, is called **real quantity**.

If we extend the realm of quantity outside of positive and negative quantity, imaginary numbers are as real as any others, as will be shown in the next article.

A **complex number** is a number part real and part imaginary; as $3+2\sqrt{-1}$ and $a+b\sqrt{-1}$.

85. Meaning of $\sqrt{-1}$.

If $OA=+1$, and OA' is of the same length, but lying in the opposite direction from O, $OA'=-1$.



Hence, we regard the operation of converting a plus quantity into negative quantity as equivalent to a rotation through an angle of 180°. If we divide this rotation into two equal rotations, each of these will be a rotation through 90°.

Hence, $\sqrt{-1}$ must be equivalent (geometrically) to the result of rotating the plus unit of quantity through 90°. Hence, $\sqrt{-1}$ on our figure will be represented by OB.

Hence, it is easy to see, also that $\sqrt{-1}\times\sqrt{-1}=-1$.

We thus perceive that the introduction of imaginary quantity enlarges the field of quantity considered in algebra from mere quantity in a line to quantity in a plane. This gives a vast extension to the power of algebraic processes and introduces many economies in them, as will be found by the student who pursues the study of mathematics extensively.

In taking up the subject for the first time, we consider only a few of the first properties of imaginaries, so called.

86. The Fundamental Principle in treating imaginaries is that $\sqrt{-1}\times\sqrt{-1}=-1$.

Using i as a symbol for $\sqrt{-1}$, this principle is, $i\times i=-1$, or $i^2=-1$.

Considering this matter algebraically, if we use the law of signs in the most general form,

$$(\sqrt{-1})^2=\sqrt{-1}\times\sqrt{-1}=\sqrt{1}=\pm 1.$$

Now, if we extract the square root of $+1$, we shall not have $\sqrt{-1}$. But if we extract the square root of -1, we shall have $\sqrt{-1}$.

Hence, we must limit the product $\sqrt{-1}\times\sqrt{-1}$ to -1.

Likewise, $\sqrt{-a}\times\sqrt{-b}=\sqrt{a}\sqrt{-1}\times\sqrt{b}\sqrt{-1}$

$$=\sqrt{a}\sqrt{b}(\sqrt{-1})^2=-\sqrt{ab}.$$

Or, using the symbol i, $ai\times bi=-ab$.

87. Operations with Imaginaries. It follows from § 86, that, in performing operations with imaginaries, we *use all the ordinary laws of algebra*, with the exception of a limitation in the use of signs, which may be mechanically stated as follows:

The product of two minus signs under the radical sign of the second degree gives a minus sign outside the radical sign. But in dividing *first indicate the division and afterwards rationalize the denominator.*

Ex. 1. Add $\sqrt{-9}$, $-3+2\sqrt{-1}$, $7-2\sqrt{-16}$.

$$\begin{array}{rl}
\sqrt{-9}= & 3\sqrt{-1}.\\
-3+2\sqrt{-1}= & -3+2\sqrt{-1}.\\
7-2\sqrt{-16}= & \underline{7-8\sqrt{-1}.}\\
 & 4-3\sqrt{-1}.\ \ \textit{Sum.}
\end{array}$$

Ex. 2. Multiply $2\sqrt{-3}+3\sqrt{-6}$ by $3\sqrt{-3}-5\sqrt{-2}$.

$$\begin{array}{l}
2\sqrt{-3}+3\sqrt{-6}\\
\underline{3\sqrt{-3}-5\sqrt{-2}}\\
6(-3)\ -9\sqrt{18}\\
\underline{\qquad\quad +10\sqrt{6}+15\sqrt{12}\qquad}\\
-18-27\sqrt{2}+10\sqrt{6}+30\sqrt{3}.\ \ \textit{Product.}
\end{array}$$

EXERCISE 72

Collect:

1. $7\sqrt{-4}+3\sqrt{-49}-10\sqrt{-9}$.
2. $5\sqrt{-\frac{1}{2}}-3\sqrt{-\frac{9}{2}}+4\sqrt{-50}-\sqrt{-200}$.

3. $a+b\sqrt{-1}-b-a\sqrt{-1}-\sqrt{-b^2}+2\sqrt{-a^2}-a.$

4. $(a-2b)\sqrt{-1}-(2a+b)\sqrt{-1}.$

5. Write $3\sqrt{-4}$, $\sqrt{-6}$, $\sqrt{-a}$, using i for $\sqrt{-1}$.

6. Simplify $5i-\frac{3}{2}i$, and use $\sqrt{-1}$ for i in the result.

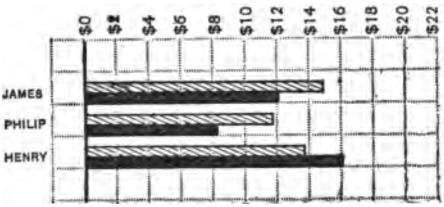
Multiply:

7. $\sqrt{-1}$ by $\sqrt{-2}$.
9. $-\sqrt{-12}$ by $-\sqrt{-18}$.
8. $\sqrt{-5}$ by $-2\sqrt{-5}$.
10. $-\sqrt{x-y}$ by $\sqrt{y-x}$.
11. $\sqrt{-1}+\sqrt{-2}$ by $\sqrt{-1}-2\sqrt{-2}$.
12. $3\sqrt{-3}-2\sqrt{-2}$ by $2\sqrt{-3}+3\sqrt{-2}$.
13. $2\sqrt{2}-2\sqrt{-2}$ by $3\sqrt{2}+3\sqrt{-2}$.
14. $3+5i$ by $5-2i$.
15. $\sqrt{-3}+\sqrt{-2}$ by $\sqrt{3}+\sqrt{-2}$.
16. $a\sqrt{-a}+b\sqrt{-b}$ by $a\sqrt{-a}-b\sqrt{-b}$.

17. Multiply $x-\dfrac{1-\sqrt{-3}}{2}$ by $x-\dfrac{1+\sqrt{-3}}{2}$.

88. Dual Bar Graphs. It is often an advantage graphically to represent two related magnitudes by two bars placed in contact.

Ex. Three boys, James, Philip, and Henry had summer gardens two years in succession. Their respective profits the first year were \$12, \$8.30, and \$16; and the second year were \$15, \$11.75, \$13.70. Represent these facts graphically by bars.



Note that on the diagram the solid bars represent the earnings of the respective boys during the first summer, and the shaded bars the earnings during the second summer.

EXERCISE 73

Make dual bar graphs for the facts given in each of the following examples:

1. The average grades of four boys, Albert, William, Howard, and Dennis for one year were 86, 94, 72, 63, respectively; and the next year were 92, 88, 82, 76.

2. The sales of thrift stamps made in one month by four girls, Anna, Harriet, Mary, and Julia were $62, $38, $56, and $84, respectively. The next month the sales made by the girls in order were $48, $54, $53, $72.

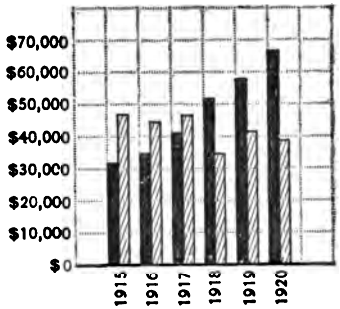
3. The profits in the five departments of a department store one year were $3285, $6924, $5872, $4296 and $9872, respectively; and the next year were $4293, $8327, $4615, $7123 and $6954, respectively.

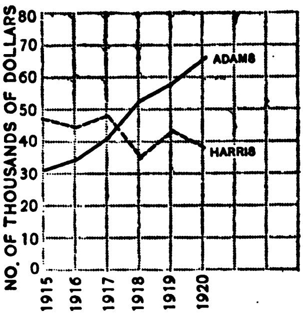
4. The average or normal rainfalls at certain representative places in the United States are as follows: Phœnix (Arizona), 7.9 in.; Denver, 14 in.; Chicago, 34 in.; New York, 44.8 in.; New Orleans, 57.4 in. During a recent year the rainfalls at these places in order were 5.4 in., 11 in., 38 in., 52 in., 48 in.

5. In a certain year the percentages of the wheat crop of the United States raised in certain states were as follows: Kansas 10.8, Nebraska 6.8, Ohio 4.5, Illinois 4.3, Indiana 4.3. The per cents of the corn crop raised by the same states in order were 4, 7, 4.8, 13, 6.5.

89. Vertical Bar Graphs. It is often an advantage to construct bar graphs in a vertical instead of a horizontal position. For instance, with vertical bars, it is frequently easier to make comparisons of a certain kind and to perceive rates of change. Also when desired (as when the facts to be represented become numerous) it is easier to substitute curves for vertical bars, than for horizontal ones.

Ex. The total sales by two salesmen, Adams and Harris, for a number of years were as follows: 1915, $31,000, $47,000; 1916, $35,000, $45,000; 1917, $41,000, $47,000; 1918, $52,000, $35,000; 1919, $58,000, $42,000; 1920, $67,000, $39,000. Graph these facts by the use of vertical bars.





In the diagram at the left these facts are represented by vertical bars.

The above data might also be represented by two curves. See the chart to the right. It is to be noted that curves showing the sales of other salesmen could also be inserted on this chart.

EXERCISE 74

In each of the following examples, represent the data by vertical bar graphs:

1. In a certain city the circulations of the five leading newspapers one year were as follows: Advertiser, 60,510; Times, 81,270; Star, 72,100; Bulletin, 91,200; Record, 36,220. The next year the circulations of these papers in order were 66,200, 91,070, 56,420, 82,760, 29,710.

2. At two towns, Melville and Dayton, the respective rainfalls in successive years were as follows: 1915, 42 in., 34 in.; 1916, 47 in., 41 in.; 1917, 39 in., 42 in.; 1918, 43 in., 39 in.; 1919, 36 in., 34 in.; 1920, 48 in., 41 in.

3. For the years specified the respective earnings of two divisions of a railroad in millions of dollars were as follows: 1915, 3.5, 6.6; 1916, 3.7, 7.2; 1917, 4.8, 6.8; 1918, 5.3, 6.6; 1919, 6.7, 5.2; 1920, 7.2, 6.5.

Also make curve line graphs of these facts. What is the advantage of the latter mode of representation?

EXERCISE 75

ORAL

1. If a ball nine has won 11 games out of 20 and is to play x more games, what is the meaning of $\frac{11}{20+x}$? Of $\frac{11+\frac{1}{2}x}{20+x}$?

2. An alloy of silver and copper weighs 40 lb., of which 7 lb. are silver. If x pounds of copper are added, what does $\frac{7}{40+x}$ mean? If y pounds of silver are then added, what does $\frac{7+y}{40+x}$ represent?

3. If a pupil has an average grade of g in three subjects and his grade in a fourth subject is x, what does $3g+x \div 4$ represent?

4. A quantity of provisions will last m men d days. How long will it last x men?

5. If sound travels 1100 ft. per second, what does $\frac{x}{1100}$ represent?

6. A rectangular piece of land is a feet long and b feet wide. If a strip w feet wide is cut from each side of the lot, what are the dimensions of the part left? What is the area? Also what is the area of the part removed?

7. A box with lid is made of boards t inches thick. If the outside dimensions of the box are a, b, c inches, what are the inside dimensions of the box? What is the volume of the inside? How many cubic feet of lumber are used in making the box and its lid? How many board feet?

8. The area of a given rectangle equals the area of a circle whose radius is r inches. If the rectangle is l inches long, what is its width?

EXERCISE 76

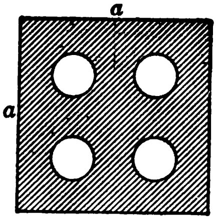


FIG. 1.

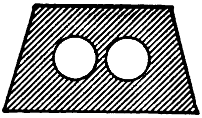


FIG. 2.

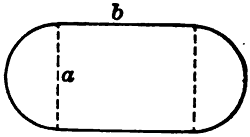


FIG. 3.

1. Fig. 1 represents a square plate whose side is a inches. From this plate four circles, the diameter of each of which is d inches, have been cut. Obtain a formula for the area that is left.

2. Fig. 2 represents a trapezoid from which two equal circles have been cut. Write a formula for the surface left, using letters selected by yourself.

3. Fig. 3 is a rectangle with semicircles attached to its ends. Obtain a formula for the area of the figure.

4. The formula for the volume of a cylinder whose radius is r and altitude h is $V=\pi r^2h$. Hence find a formula for the number of tons of silage in a cylindrical silo if 40 cu. ft. make one ton.

By use of this formula, find the radius r of a silo which is to contain 80 tons of silage and be 16 ft. high.

5. The width of a cold-air box to a furnace is w inches. Find a formula for its depth (d) in inches if the area of the cross-section of the box is to equal $\frac{3}{4}$ the area of the sum of a circles each b inches in diameter and c circles each d inches in diameter.

6. A wheel d inches in diameter is revolving n times a second. Obtain a formula for the velocity in feet per second (v) of a point on the rim of the wheel, using $\pi=\frac{22}{7}$.

Solve the formula for n. What is the use of this last result? Illustrate the use of the last formula by a numerical example.

7. If a steel bar expands .000007 of its length for each degree in the increase of its temperature (Fahrenheit), find a formula for the increase in feet (I) in the length of a steel rail l feet long, when the temperature increases from t to t' degrees. (In the formula write .000007 in a short way by the use of a negative exponent. See p. 114.)

Find I when $l = 30$, $t = 30$, $t' = 132$.

8. Two cars start at the same time and place and travel, one due north at the rate of a miles per hour, and the other due east at the rate of b miles per hour. Find a formula for their distance apart in miles (D) at the end of t hours.

Illustrate the use of your formula by a numerical example.

9. Successive discounts of 40 % and 5 % were given on an article listed at $250. Find the net cost of the article.

Now obtain a formula for the net cost (N) of an article whose list price (L) is subject to two successive discounts, d_1 and d_2 (expressed decimally).

By use of your formula, show that it is immaterial which of these two discounts is made first.

10. If a adults were admitted to a ball game at 50 cents each, and c children at 25 cents each, obtain a formula for T, the total receipts in dollars.

If, in a given case, the total receipts and the number of adults admitted are known, how can the number of children be determined by use of the formula? Illustrate by a numerical example.

11. If a long ton contains 2240 lb. and a short ton 2000 lb., obtain a formula for converting long tons into short tons.

Illustrate the use of your formula by a numerical example.

CHAPTER VII

QUADRATIC EQUATIONS; SIMULTANEOUS QUADRATICS

90. A Quadratic Equation of one unknown quantity is an equation containing the second power of the unknown, but no higher power.

Ex. $3x^2-5x+7=0.$

What is a pure quadratic equation? An affected quadratic equation?

Solve $x^2=9$. Also $4x^2-25=0$.

91. Completing the Square.

Ex. Solve $6x^2-11x=10$ by completing the square.

Dividing by 6, $x^2-\frac{11}{6}x=\frac{5}{3}.$

Taking half the coefficient of x, that is, $\frac{11}{12}$, squaring this, and adding the result to each side of the equation, we obtain

$$x^2-\frac{11}{6}x+\frac{121}{144}=\frac{361}{144}.$$

Hence

$$x-\frac{11}{12}=\pm\frac{19}{12}.$$

$$x=\frac{11}{12}\pm\frac{19}{12}$$

$$x=\frac{5}{2},\ -\frac{2}{3}.\quad \textit{Ans.}$$

CHECK for $x=\frac{5}{2}$.	CHECK for $x=-\frac{2}{3}$.
$6x^2-11x=6(\frac{25}{4})-11(\frac{5}{2})=10.$	$6x^2-11x=6(\frac{4}{9})+11(\frac{2}{3})=10.$

Hence, we have the general rule:

By clearing the given equation of fractions and parentheses, transposing terms, and dividing by the coefficient of x^2, *reduce the given equation to the form* $x^2+px=q$;

Add the square of half the coefficient of x *to each member of the equation;*

Extract the square root of each member;

Solve the two resulting simple equations.

Before clearing a given equation of fractions, it is important to *reduce each fraction in the given equation to its simplest form.*

92. Use of Formula. Any quadratic equation can be reduced to the form

$$ax^2+bx+c=0.$$

Solving this equation by completing the square (§ 91),

$$x=\frac{-b\pm\sqrt{b^2-4\,ac}}{2\,a}.$$

By substituting in this result, as a formula, the values of a, b, c in any given equation, the value of x may be obtained.

Ex. Solve $3\,x^2-4\,x-7=0$ by use of the formula.

Here $a=3,\quad b=-4,\quad c=-7.$

Substituting for a, b, c in the above formula.

$$x=\frac{4\pm\sqrt{16+84}}{6}=\frac{4\pm10}{6}=\frac{7}{3},\quad -1.\quad \textit{Roots.}$$

Let the pupil check the work.

93. Literal Quadratic Equations are solved by the methods employed in solving quadratic equations with numerical coefficients.

Ex. Solve $ax+1=\dfrac{3-a^2x^2}{ax}.$

Clearing of fractions, $a^2x^2+ax=3-a^2x^2.$

Hence, $2\,a^2x^2+ax=3.$

$$x^2+\frac{x}{2\,a}=\frac{3}{2\,a^2}.$$

$$x^2+\frac{x}{2\,a}+\left(\frac{1}{4\,a}\right)^2=\frac{25}{16\,a^2}.$$

$$x=\frac{1}{a},\quad -\frac{3}{2\,a}.\quad \textit{Roots.}$$

Let the pupil check the work.

EXERCISE 77

Solve and check:

1. $x^2-7x=-12.$

2. $6x^2-7x-5=0.$

3. $x^2+8\frac{1}{6}x+10\frac{5}{6}=0.$

4. $x^2-.5x+.06=0.$

5. $6x^2=13x(x+1)-6(x+1)^2.$

6. $\dfrac{2x+3}{2(2x-1)}-\dfrac{7-x}{2(x+1)}=\dfrac{7-3x}{4-3x}.$

7. $\dfrac{x-1}{2}=\dfrac{5}{2}-\dfrac{2}{x-1}.$

8. $\dfrac{3x-4}{x-1}-\dfrac{4x-14}{1-x^2}=\dfrac{2x-1}{x+1}-1.$

9. $\frac{1}{3}(13-5x)-\frac{1}{2}(x^2-x)=0.$

Find the value of x to three decimal places in

10. $x^2+5x-7=0.$

11. $3x^2+5x-7=0.$

Solve and check:

12. $x^2-3cx=10c^2.$

13. $2x^2+px=10p^2.$

14. $3c^2x^2=4cdx+4d^2.$

15. $x^2=\dfrac{15m^2}{8a^2}-\dfrac{mx}{4a}.$

16. $x^2-(a-b)x=ab.$

17. $x^2+3ax+2bx=-6ab.$

18. $\dfrac{2a+x}{2a-x}+\dfrac{a-2x}{a+2x}=\dfrac{8}{3}.$

19. $ax^2+px+q=0.$

20. $\dfrac{x^2-3ax}{a-b}+2a=\dfrac{bx}{b-a}.$

21. $x^2+bx+c=0.$

22. $\dfrac{a-2b}{8x^2-2b^2}=\dfrac{1}{2x+b}-\dfrac{1}{2a}.$

23. Solve $V=.97\sqrt{2gH}$ for H.

24. Solve $d=\sqrt{\dfrac{3l(L-l)}{8}}$ for l.

25. Solve $v=\frac{1}{3}\pi h(R^2+r^2+Rr)$ for R.

26. In $p=6r-r^2$, find r to two decimal places when $p=1$. Also when $p=2$; $\frac{1}{4}$. Is anything gained by solving for r before substituting numerical values for p? Give reasons for your answer.

94. The Factorial Method of solving equations consists in transposing all terms to the left-hand side, factoring the resulting expression, and letting each factor $=0$.

Ex. 1. Solve $x^3+8=0$.

Factoring, $(x+2)(x^2-2x+4)=0$,

$x+2=0$, gives $x=-2$. *Root.*

Also, $x^2-2x+4=0$.

Whence. $x^2-2x=-4$.

$x=1\pm\sqrt{-3}$. *Roots.*

Let the pupil check the work.

The factorial method of solution is especially helpful in solving certain literal quadratic equations.

Ex. 2. Solve $(p+q)x^2-(2p+q)x+p=0$ by the factorial method.

We obtain $[(p+q)x-p](x-1)=0$.

Hence, $x=1, \frac{p}{p+q}$. *Roots.*

Let the pupil check the work.

If this example be solved by the method of completing the square (§ 91), on comparing the two solutions, it will be found that at least three fourths of the labor of solution is saved by use of the factorial method.

EXERCISE 78

Solve and check:

1. $x^2-6x+8=0$.
2. $x^3-6x^2+8x=0$.
3. $2x^2+5x-12=0$.
4. $3x^2+3=10x$.

5. $x(x-1)(x+2)(x-3)=0.$

6. $0=3x(x^2-4).$

7. $x^3-27=0.$

8. $x^3=27.$

9. $x^5=16x.$

10. $x^4-5a^2x^2+4a^4=0.$

11. $x^5+x^4-x-1=0.$

12. $3(x^2-4)=5(x-2).$

13. Solve $x^2-(2a+5b)x+10ab=0$ by the method of § 91. Also solve the same equation by the factorial method. Compare the amount of work in the two processes. Why do we not solve all quadratic equations by the factorial method?

Solve by the factorial method:

14. $x^2+3ax-10a^2=0.$

15. $x^2+2x-3bx=6b.$

16. $cdx^2+c^2x+d^2x+cd=0.$

17. $\dfrac{x+1}{x^2}=\dfrac{p+1}{p^2}.$

18. $\dfrac{x}{b-x}+\dfrac{b+c}{x}=\dfrac{b}{b-x}.$

95. Equations in the Quadratic Form. An equation containing only two powers of the unknown quantity, the index of one power being twice the index of the other power, is an equation of the quadratic form. It may be solved by the methods already given for affected quadratic equations.

Ex. 1. Solve $x^4-10x^2=-9.$

Adding 5^2 to both members will make the left-hand member a perfect square. Thus,

$$x^4-10x^2+25=16.$$

Hence, $x^2-5=\pm 4.$

$$x^2=9, \text{ or } 1.$$

$$x=\pm 3,\ \pm 1. \quad \textit{Roots.}$$

Let the pupil check the work.

This equation might also have been solved by the factorial method.

Ex. 2. Solve $x^{\frac{2}{5}}-3x^{\frac{1}{5}}+2=0$.

Hence, $(x^{\frac{1}{5}}-1)(x^{\frac{1}{5}}-2)=0.$

$$x^{\frac{1}{5}}=1,\ 2.$$

$$x=1,\ 32.\quad \textit{Ans.}$$

EXERCISE 79

Solve and check:

1. $x^4-3x^2+2=0.$
2. $x^4-26x^2=-25.$
3. $13x^2=9x^4+4.$
4. $9x^4+1=10x^2.$
5. $x^6-9x^3+8=0.$
6. $x^6-26x^3-27=0.$
7. $x^{\frac{2}{3}}-5x^{\frac{1}{3}}+6=0.$
8. $x^{\frac{1}{2}}+x^{\frac{1}{4}}-2=0.$
9. $3x^{\frac{1}{2}}-5x^{\frac{1}{4}}=2.$
10. $3x^{\frac{2}{3}}=4x^{\frac{1}{3}}+4.$
11. $2\sqrt[3]{x}=\sqrt[6]{x}+1.$
12. $x^3-28x^{\frac{3}{2}}+27=0.$
13. $x^5-33x^{\frac{5}{2}}+32=0.$

14. Find the roots of $x^4-5x^2+3=0$ to the nearest hundredth.

15. Is $3-\sqrt{5}$ a root of $x^4-7x^2+5=0$?

16. Is $\frac{1}{2}(2-\sqrt{3})$ a root of $x^2-2x+\frac{1}{4}=0$?

96. Radical Equations Resulting in Affected Quadratic Equations. If an equation is cleared of radicals, the result is often a quadratic equation.

Ex. Solve $\sqrt{x+5}-\sqrt{2x-7}=\sqrt{x}$.

Transposing, $\sqrt{x+5}-\sqrt{x}=\sqrt{2x-7}.$

Squaring, $x+5-2\sqrt{x^2+5x}+x=2x-7.$

Hence, $\sqrt{x^2+5x}=6.$

Whence, $x^2+5x=36.$

$$x=4,\ -9.$$

Substituting these values in the original equation, we find that the only value that verifies is $x=4$, which is the root. The other value, $x=-9$, is not a root of the original equation, but is introduced by squaring in the process of clearing the equation of radical signs. It satisfies the equation,

$$\sqrt{x+5}-\sqrt{2x-7}=-\sqrt{x}.$$

(See the treatment of extraneous roots, p. 133.)

EXERCISE 80

Solve and check:

1. $x+5\sqrt{x+2}-4=0.$
2. $4=\sqrt{4x+17}+\sqrt{x+1}.$
3. $\sqrt{2+x}+\sqrt{2-x}=\sqrt{3}.$
4. $\sqrt{2x+5}+\sqrt{3x+4}=\sqrt{5x+9}.$
5. $\sqrt{2x+7}+\sqrt{3x-18}-\sqrt{7x+1}=0.$
6. $\sqrt{1-x}+\sqrt{1-x-\sqrt{x}}=1.$
7. $\sqrt{a-x}+\sqrt{a+x}-\sqrt{2a+2b}=0.$
8. $\sqrt{3x+1}-2\sqrt{2x+1}+\dfrac{3}{\sqrt{2x+1}}=0.$
9. $\dfrac{3}{2+\sqrt{12-x}}-\dfrac{\sqrt{12-x}}{5}=0.$
10. $\dfrac{\sqrt{x}}{\sqrt{x+2}}-\dfrac{\sqrt{x+2}}{\sqrt{x}}-\dfrac{5}{6}=0.$

EXERCISE 81

REVIEW

Solve and check:

1. $12x^2+x=1.$
2. $10a^2=x^2-3ax.$
3. $2\sqrt{x+3}-\sqrt{x-2}=4.$
4. $x^4-8x=0.$
5. $x^4-6x^2+8=0.$
6. $x^3-9x^{\frac{3}{2}}+8=0.$

7. $10x^2-.7x-.12=0.$

8. $x^2+3bx=6ab+2ax.$

9. $x^{\frac{2}{3}}-3x^{\frac{1}{3}}+2=0.$

10. $2\sqrt{x-1}=5-\frac{2}{\sqrt{x-1}}.$

11. $x^7-x=0.$

12. $x+\frac{1}{x}=\frac{a}{b}+\frac{b}{a}.$

13. $\frac{15}{4}-\frac{x}{5-x}=\frac{x-5}{x}.$

14. $x(x-2)(9x^2-25)=0.$

15. $\sqrt{3x+10}=\sqrt{10x+16}+\sqrt{x+2}.$

16. $2x^2-5.2-3.7=0$ (to nearest hundredth).

17. $\sqrt{2x-a^2}+\sqrt{x}-\sqrt{x+3a^2}=0.$

18. Write but do not solve an equation of each of the principal kinds treated thus far in this chapter.

EXERCISE 82

Oral

1. If a man rows at the rate of x miles an hour on a stream which flows y miles per hour, what is denoted by $x+y$? By $\frac{1}{2}x+y$? By $\frac{2}{3}x-y$?

2. A certain calf eats 3 bushels of grain while a cow eats 4 bushels. During a certain length of time, they together ate b bushels. How many bushels did each eat?

3. The minuend is $m+2n$, the remainder is $2n+s$. What is the subtrahend?

4. If 40 bu. of oats worth a cents a bushel, and 30 bu. of corn worth b cents a bushel are mixed, what does $\frac{40a+30b}{70}$ represent?

5. If one pipe can fill a given cistern in f hours, and another can fill it in g hours, what is the meaning of $\frac{1}{f}+\frac{1}{g}$? Of $\frac{x}{f}-\frac{y}{g}$?

6. The dividend in a certain process is $a-x$, the quotient is a, and the remainder is 4. What is the divisor?

7. A mixture of equal parts of two kinds of tea, costing a and b cents a pound respectively, is sold at a gain of 25 %. Find the selling price of the mixture per pound?

8. A man travels x hours at the rate of m miles per hour and then finds he still has $\frac{5}{8}$ of his journey to go. How far did he intend to go?

9. A rectangular lot is x yards wide and $2x$ yards long. If a strip 5 yd. wide be added on all sides of the rectangle, what will be the length and width of the new rectangle?

10. A sheet of tin is a inches long, and b inches wide. If a square whose side is 5 in. be cut out of each corner, and the sides be folded up so as to make a box, what will the depth of the box be? The area of its bottom? Its volume?

EXERCISE 83

1. Three times the square of a given number, diminished by twice the product of the number and the next lower number, gives 143. Find the number.

2. A certain number increased by four times its reciprocal equals $8\frac{1}{2}$. Find the number.

3. The denominator of a given fraction exceeds the numerator by 3, and the sum of the fraction and its reciprocal is $2\frac{9}{40}$. Find the fraction.

4. The base of a triangle is 4 ft. less than the altitude, and the area of the triangle is 48 sq. ft. Find the base.

5. A rectangular lot is surrounded on all sides by a driveway 5 yd. wide. The lot is twice as long as it is wide. If the area of the lot and driveway together is 6600 sq. yd., find the dimensions of the lot.

6. A farmer has a field 60 rd. long and 40 rd. wide. How wide a strip must he cut around the field in order that 5 acres may be left uncut?

7. An open box is to be formed by cutting out equal squares from the corners of a square sheet of tin and folding up the sides. The box is to be 10 in. deep and is to contain 2250 cu. in. Find the length of a side of the square sheet of tin.

8. One baseball nine has won 6 games out of 15, and another has won 8 out of 13. How many straight games must the first nine win from the second, in order that the average of games won by the nines shall be the same?

9. A and B together can do a given piece of work in 2 days. Working alone, A could do the work in 3 days less than B. How many days would it take each man working alone?

10. Two pipes together can fill a tank in $3\frac{1}{5}$ hr. When running alone, one pipe takes 12 hr. longer than the other to fill the tank. How long does it take each pipe alone to fill it?

11. A rectangular piece of tin is 12 in. long and 9 in. wide. Four equal squares are cut out, one from each corner so that when the sides are folded up a box is made the area of whose bottom is 60 sq. in. Find an edge of one of the small squares cut out.

12. The area of a mat about a picture 10 in. long and 8 in. wide is one half the area of the picture. What are the outside dimensions of the mat?

13. The altitude of a right triangle is 1 in. greater than the base, and the hypotenuse is 2 in. greater than the base. Find the sides of the triangle.

14. The perimeters of a given circle, square, and rectangle are each 160 in., the rectangle is three times as long as it is wide. Find the area of each figure. What geometric principle is illustrated by the results obtained?

Draw the three figures, using the same scale for all.

15. The side of a given square is a feet. By how many feet must this side be increased in order that the area of the square may be increased by b square feet?

16. A given field of grain contains 20 acres and is twice as long as it is wide. How wide a strip (in rods) must be cut around the field, in order that $\frac{5}{8}$ of the grain shall be left uncut?

17. A coal bin is to be 8 ft. deep and three times as long as it is wide, and is to contain 15 tons of coal. If 40 cu. ft. is allowed for 1 ton, how long must the bin be?

18. One leg of a given triangle exceeds the other by 2 ft. If the hypotenuse is 10 ft., find the legs of the triangle.

19. Find the side of a square whose diagonal is a inches.

20. The product of two consecutive numbers is b. Find the numbers.

97. Quadratic Equations Containing Two Unknowns. The general quadratic equation containing two unknowns is

$$ax^2+bxy+cy^2+dx+ey+f=0.$$

By giving a, b, c, etc., different numerical values, including zero, this general equation may be made to take many special forms.

What values must we give $a, b, c \ldots$, respectively, in order to obtain the equation $5x^2+3xy+2y^2=5$ from the general equation?

The **absolute term** of an equation is the term which does not contain an unknown factor. Thus f in the above general equation is the absolute term.

Simultaneous quadratic equations is a brief term for simultaneous equations whose solution involves quadratic equations.

Thus, equations (1) and (2) in § 99 are simultaneous quadratic equations.

In general, the combination of two simultaneous quadratic equations by elimination gives an equation of the fourth degree in one unknown, which cannot be solved by the methods taught in this book. Two simultaneous quadratic equations can be solved by elementary methods only in certain special cases.

98. A Homogeneous Equation is one in which all the terms containing an unknown quantity are of the same degree.

Thus, $3x^2y-5xy^2+y^3=18$ is a homogeneous equation of the third degree. What is the degree of the equation $xy=6$?

CASE I

99. When One Equation is of the First Degree and the Other of the Second, two simultaneous equations *may always be solved by the method of substitution.*

Ex. Solve
$$\begin{cases} 2x-3y=3. & \dots\dots\dots (1) \\ 4x^2-7xy=15. & \dots\dots\dots (2) \end{cases}$$

Eliminate y, since y occurs only once in equation (2).

From (1),
$$y=\frac{2x-3}{3}. \quad \dots\dots \quad (3)$$

Hence, from (2), $4x^2-7x\left(\frac{2x-3}{3}\right)=15.$

Hence, $12x^2-14x^2+21x=45.$

Substitute for x in (3),
$$\left.\begin{array}{l} x=3,\ \frac{15}{2}. \\ y=1,\ 4. \end{array}\right\} \textit{Roots.}$$

CHECK.	CHECK.
For $x=3$ and $y=1$.	For $x=\frac{15}{2}$ and $y=4$.
$2x-3y=6-3=3.$	$2x-3y=15-12=3.$
$4x^2-7xy=36-21=15.$	$4x^2-7xy=225-210=15.$

EXERCISE 84

Solve and check:

1. $3x-1=y.$
$5x^2-y^2=1.$

2. $x+2y=1.$
$x^2+4y^2=6x+11.$

3. $2x^2+3xy-4y^2+3x-8y=14.$
$7x-5y=14.$

4. $x^2-4y^2-3x=0.$
$2+\frac{5y}{x}-\frac{3}{x}=0.$

5. $2x+3y=1.$
$\frac{6}{x}+\frac{1}{y}=2.$

6. $\frac{x}{2}+\frac{y}{3}=2.$
$\frac{2}{x}+\frac{3}{y}=2.$

7. $y-4=\frac{1}{2}(y-x).$
$xy=2y+x+2.$

8. $5x-3y+1=0.$
$2y^2+3xy-5x^2+7x-6y=4.$

9. $xy=36.$
$(x-3)(y+1)=30.$

10. $xy=180.$
$(x+6)(y-1)=180.$

Without solving the equations, determine whether

11. $x=1$ and $y=3$ satisfy $\begin{cases} 3x-2y=9. \\ 2x^2+10-3y^2=5xy. \end{cases}$

12. $x=2$, $y=-\frac{8}{3}$ are roots of $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=\frac{1}{3}. \\ (x-y)^2=y^2-7. \end{cases}$

13. The sum of two numbers is 3, and twice the square of the second number diminished by five times the square of the first number gives 3. Find the two numbers.

[illegible] the
[illegible] term.

CASE II

100. When Both Equations are Homogeneous and of the Second Degree, two simultaneous quadratic equations *may always be solved by eliminating the absolute term between* *them and factoring to find the value of one unknown in terms of the other, and then proceeding as in Case I.*

Ex. Solve $2x^2-3xy+4y^2=6$. . . . . (1)

$x^2+3y^2=7$. . . . . (2)

Multiply (1) by 7 and (2) by 6.

Hence, $14x^2-21xy+28y^2=42$,

and $6x^2+18y^2=42$.

$\therefore 14x^2-21xy+28y^2=6x^2+18y^2$,

or, $8x^2-21xy+10y^2=0$.

$\therefore (8x-5y)(x-2y)=0$.

$x-2y=0$.	$8x-5y=0$.
$x^2+3y^2=7$.	$x^2+3y^2=7$.
Hence, solving by Case I,	Hence, solving by Case I,
$x=\pm 2$,	$x=\pm\frac{5}{31}\sqrt{31}$.
$y=\pm 1$.	$y=\pm\frac{8}{31}\sqrt{31}$.

Hence, $\left.\begin{array}{l} x=\pm 2,\ \pm\sqrt{31}. \\ y=\pm 1,\ \pm\sqrt{31}. \end{array}\right\}$ *Roots.*

Let the pupil check the work.

EXERCISE 85

Solve and check:

1. $y^2+3xy=28$.
$4x^2+xy=8$.

2. $4x^2-3xy+2=0$.
$3x^2-2y^2+6=0$.

3. $y^2=5+2xy$.
$x^2+y^2=29$.

4. $y^2+xy=21$.
$2xy-x^2=8$.

5. $x^2-xy+y^2=21$.
$x^2-2xy=-15$.

6. $2x^2-4xy+3y^2=17$.
$x^2-y^2=16$.

7. $x^2+xy+y^2=1$.
$2x^2+3xy+4y^2=3$.

8. $4x^2+2xy-y^2=41$.
$8x^2-5xy+y^2=58$.

Solve and check the following miscellaneous examples:

9. $2x = y - 3.$
$4x^2 + y^2 = 17.$

10. $x^2 + 3y^2 = 12.$
$5xy - 4y^2 = 11.$

11. $\frac{2y}{x} = \frac{x+y}{y}.$
$x - y - a = 0.$

12. $46 + x^2 = 2y^2.$
$x^2 = 14 - xy.$

13. $\frac{1}{3}(x - 1) = \frac{1}{2}y.$
$y^2 - 7 = (x - y)^2.$

14. $4y^2 - xy = x^2 - 16.$
$1 = \frac{28}{x^2} - \frac{3y}{x}.$

15. Point out the examples in Exercise 88 (p. 162), which come under Case I. Under Case II.

16. State Ex. 1 as a problem concerning two numbers.

17. Find a number consisting of two digits such that if the number is multiplied by the left-hand digit, the result will be 105; but if the number is multiplied by the right-hand digit, the result will be 175.

101. The Methods of Cases I and II are the **only general methods** which can be used in solving all simultaneous quadratic equations of a given class. Besides these, however, there are certain special methods which enable us to solve important particular examples.

Examples which come directly under Cases I and II are often solved more advantageously by one of these special methods.

The special methods apply with particular advantage to symmetrical equations.

102. A Symmetrical Equation is one in which, if y is substituted for x, and x for y, the resulting equation is identical with the original equation.

Thus, each of the following is a symmetrical equation:

$$x^3 + 3x^2y^2 + y^3 = 18, \quad x + y = 12, \quad xy = 6.$$

CASE III

103. Addition and Subtraction Method (often in connection with multiplication and division). In this method the object is to find *first, the values of* $x+y$ *and* $x-y$, *and then the values of* x *and* y *themselves.*

Ex. 1. Solve $\begin{cases} x+y=7. & \dots\dots\dots (1) \\ xy=10. & \dots\dots\dots (2) \end{cases}$

Here we have the value of $x+y$ given, and the first object is to find the value of $x-y$.

Square (1), $x^2+2xy+y^2=49.$ . . . . . . (3)

Multiply (2) by 4, $4xy=40.$ . . . . . . (4)

Subtract (4) from (3), $x^2-2xy+y^2=9.$ . . . . . . (5)

Extract square root of (5), $x-y=\pm 3.$ . . . . . . (6)

Add (1) and (6), divide by 2, $x=5, 2.$
Subtract (6) from (1), divide by 2, $y=2, 5.$ } *Roots.*

Let the pupil check the work.

Ex. 2. Solve $\begin{cases} x^3+y^3=28. & \dots\dots\dots (1) \\ x+y=4. & \dots\dots\dots (2) \end{cases}$

Divide (1) by (2), $x^2-xy+y^2=7.$ . . . . (3)

Square (2), $x^2+2xy+y^2=16.$ . . . . (4)

Subtract (3) from (4), $3xy=9.$

Hence, $xy=3.$ . . . . (5)

Subtract (5) from (3), $x^2-2xy+y^2=4.$

$\therefore x-y=\pm 2.$

But $x+y=4.$

Hence, $\left.\begin{matrix} x=3, 1. \\ y=1, 3. \end{matrix}\right\}$ *Roots.*

Ex. 3. Solve $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=7. & (1) \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=25. & (2) \end{cases}$

Squaring (1), $\dfrac{1}{x^2}+\dfrac{2}{xy}+\dfrac{1}{y^2}=49$ . . . . (3)

Subtracting (2) from (3), $\dfrac{2}{xy}=24.$ . . . . (4)

Subtracting (4) from (2), $\dfrac{1}{x^2}-\dfrac{2}{xy}+\dfrac{1}{y^2}=1$ . . . . (5)

Hence, $\dfrac{1}{x}-\dfrac{1}{y}=\pm 1.$

But, from (1), $\dfrac{1}{x}+\dfrac{1}{y}=7.$

Hence, adding, $\dfrac{2}{x}=8,\ 6.$

$\left.\begin{aligned} \therefore x&=\tfrac{1}{4},\ \tfrac{1}{3}. \\ y&=\tfrac{1}{3},\ \tfrac{1}{4}. \end{aligned}\right\}$ Roots.

Let the pupil check the work.

EXERCISE 56

Solve and check:

1. $x+y=2.$
   $xy=-15.$
2. $x+y=5.$
   $x^2+y^2=13.$
3. $x^2+y^2=34.$
   $xy=-15.$
4. $x^2+xy+y^2=28.$
   $x+y-6=0.$
5. $9(x^2+y^2)=37.$
   $xy=-\frac{2}{3}.$
6. $x^2+y^2=26a^2.$
   $x+y=4a.$
7. $x^2+y^2=3\frac{1}{2}.$
   $x+y=2.$
8. $x^2+y^2=37.$
   $x^2-xy+y^2=37.$
9. $x^2+y^2=2a^2+2b$
   $x+y=2a.$
10. $x^4+y^4=17.$
    $xy=2.$

11. $x^4+y^4=17.$
$xy=-2.$

12. $x^4+y^4=257.$
$xy=4.$

13. $\frac{1}{x}+\frac{1}{y}=7.$
$\frac{1}{xy}=12.$

14. $\frac{1}{x^3}+\frac{1}{y^3}=9.$
$\frac{1}{x}+\frac{1}{y}=3.$

15. $x^4+x^2y^2+y^4=21.$
$x^2+xy+y^2=7.$

16. $x^2+xy+y^2=133.$
$x-\sqrt{xy}+y=7.$

17. $xy=80.$
$\frac{1}{x}-\frac{1}{y}=\frac{1}{5}.$

18. $x^2+y^2=25.$
$x-y=7.$

19. $x^3-y^3=6\,a^2b+2\,b^3.$
$x-y=2\,b.$

EXERCISE 87

1. Eliminate p between $p=by$, and $p^2+y^2=c^2$.

2. Eliminate c between $y=cd^2$, and $x=cf^2$.

3. Eliminate a between $x-3\,a=7$, and $x^2+y^2=6\,a^2$.

4. If $K=\pi R^2$ and $C=2\,\pi R$, find K when $C=10$ and $\pi=\frac{22}{7}$.

5. If $s=\frac{1}{2}\,gt^2$ and $v=gt$, find s when $g=32$ and $v=64$.

6. If $C=2\,\pi R$ and $V=\frac{4}{3}\,\pi R^3$, find V when $C=33$ and $\pi=\frac{22}{7}$. (Use cancellation wherever possible.)

7. Given $l=a+(n-1)d$ and $s=\frac{n}{2}(a+l)$, find s in terms of d, n, and l.

SUG. What letter must be eliminated?

8. Eliminate l between $l=ar^{n-1}$ and $s=\frac{rl-a}{r-1}$.

9. Given $S=\pi RL$ and $T=\pi R(R+L)$, eliminate R and find T in terms of S and L.

10. Given $S=4\pi R^2$, and $V=\frac{4}{3}\pi R^3$, eliminate R and find V in terms of S.

11. By finding the value of t in the first equation and substituting in the second, eliminate t between the equations $v=u+gt$ and $s=ut+\frac{1}{2}gt^2$. Hence find s when $g=32$, $v=10.4$, and $u=2.2$.

EXERCISE 88

REVIEW

Solve and check:

1. $2x-5y=0.$
$x^2-3y^2=13.$

2. $x+y=2.$
$\frac{2}{x}+\frac{3}{y}=6.$

3. $2x^2-xy=28.$
$x^2+2y^2=18.$

4. $x^3+y^3=91.$
$x+y=1.$

5. $\frac{1}{x}+\frac{1}{y}=13.$
$\frac{1}{xy}=36.$

6. $x^2+3y^2=28.$
$x^2+xy+2y^2=16.$

7. $\frac{2x}{y}=\frac{x+y}{x}.$

8. $x^4+y^4=17.$
$x+y=3.$

9. $x^2+2x-y=5.$
$2x^2-3x+2y=8.$

10. $xy=42.$
$(x+3)(y-2)=40.$

11. $x+y=5\sqrt{5}.$
$x^2-xy+y^2=35.$

12. $x^4+x^2y^2+y^4=91.$
$x^2+xy+y^2=7.$

13. $x+y=4.7.$
$xy=4.8.$

14. $x^{-1}+y^{-1}=2.$
$x^{-2}+y^{-2}=2\frac{1}{2}.$

15. $x^{-2}+y^{-2}=13.$
$6xy=1.$

16. $11=a+2(n-1).$
$36=\frac{n}{2}(a+11).$

17. $x^{\frac{1}{3}}-y^{\frac{1}{3}}=2.$
$x-y=26.$

18. $\frac{1}{x}-\frac{2}{y}=3,\ \frac{1}{x^2}+\frac{4}{y^2}=17.$

EXERCISE 89

ORAL

1. If a rectangle is x yards long and y yards wide and a strip 5 yd. wide is added on each side, what will be the area of the new rectangle thus formed? What will be the length of its diagonal?

2. A given quantity of provisions will last a men b days. How long will it last if c men join the a men after 3 days?

3. If an automobile travels at the rate of x miles per hour, how far will it go in y hours and 45 minutes?

4. If the outside diameter of a pipe is d inches, and the pipe is t inches thick, express the radius of the hollow interior of the pipe. Also the area of a cross-section of this interior.

5. If one gallon of paint will cover 200 sq. ft. of surface, how many gallons are needed in all for two walls each l by h feet and two other walls each w by h feet?

6. How many gallons are needed for a trapezoid whose parallel sides are a and b feet and whose altitude is h feet?

7. A box (without lid) is made of boards t inches thick. If the outside dimensions of the box are a, b, c inchs, c being the depth, express the inside dimensions of the box.

8. How many bushels are in a bin whose dimensions are a, b, c feet, if 1 bushel $= 1\frac{1}{4}$ cu. ft.?

9. If water is running through a rectangular opening a feet wide and d feet deep at the rate of f feet per minute, how many cubic feet will run through in m minutes?

EXERCISE 90

1. Find two numbers whose sum is 11 and the sum of whose cubes is 539.

2. The difference of the cubes of two numbers is 279. Also the sum of the squares of the numbers increased by their product is 93. Find the numbers.

3. The sum of two numbers increased by twice their product equals 85. Also five times the smaller exceeds twice the larger by 2. Find the numbers.

4. The hypotenuse of a right triangle is 20, and the sum of the other two sides is 28. Find the length of the sides.

5. The diagonal of a rectangle is equal to the width increased by two thirds of the length. The area is 60 sq. ft. Find the dimensions of the rectangle.

6. The area of a single tennis court is 234 sq. yd. If a margin 15 ft. wide is added on all sides, the area is 684 sq. yd. Find the dimensions of the court.

7. A rectangular piece of tin is made into an open box by cutting a 5-inch square from each corner and turning up the sides and ends. If a 3-inch square were cut from each corner, the box, made in the same way, would hold the same amount. The width of the tin is 4 inches less than its length. Find the volume of the box in cubic inches.

8. If a baseball nine should play three more games and win them all, its average of games won would be $\frac{2}{3}$. But if it should play 12 more games and win them all, its present average of games won would be increased by 20 per cent of itself. Find the number of games it has already played and the number it has won.

9. A laborer received $15 for a certain number of days' work. If he had received 25 cents less a day, it would have taken him two days longer to earn the same amount. How long did he work?

10. Two trains run 36 miles at uniform rates. One train travels 9 miles per hour faster than the other and makes the trip in 12 minutes less than the other. Find the rates of the trains.

11. A rectangular court can be paved with 288 square tiles of a certain size. But if a side of the tile used is increased by 3 in., only 162 tiles are needed. Find a side of each kind of tile.

12. A certain number contains two digits. If the number is multiplied by the units' digit, the product is 216. But if the number is multiplied by the number obtained by reversing the order of the digits, the product is 2268. Find the number.

13. Telegraph poles are set at equal distances apart. In order to have two less to the mile, it will be necessary to set them 20 ft. farther apart. Find how far apart they are now.

14. A rectangular field of grain contains 30 acres. If a strip 10 rd. wide is cut around the field, 15 acres will be left uncut. Find the dimension of the field.

15. The denominator of a certain fraction exceeds the numerator by 6. If the numerator of the fraction is increased by 3 and the denominator by 15, the value of the fraction becomes $\frac{2}{3}$ of what it was originally. Find the fraction.

16. The product of two numbers is 85. If the larger of the two numbers is divided by the smaller, the quotient is 3 and the remainder 2. Find the numbers.

17. A company contracted to make 252 automobiles. Two factories working together, can make this number in 12 days. Working alone, one factory requires 7 days longer than the other to do this amount. Find the number of days in which each factory alone can fulfill the contract.

18. Find two numbers whose sum is b and the sum of whose cubes is a.

19. The area of a right triangle is p and the hypotenuse is q. Find the other two sides.

EXERCISE 91

1. If a fence is to be 120 ft. long and the posts are 10 ft. apart, how many posts will be needed?

2. If a fence is to be a feet long and the posts b feet apart, and b is an exact divisor of a, how many posts will be needed?

3. If a fence is put around a lot l feet square, and the posts are f feet apart, and f is an exact divisor of l, obtain a formula for N, the number of posts needed.

4. Make up and solve a similar problem concerning a fence about a rectangular field.

5. Obtain a formula for I, the number of cubic inches of iron used in making a pipe l feet long, whose outside diameter is d inches and in which the iron is t inches thick.

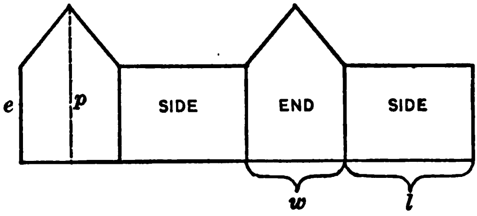
6. If two successive discounts d_1 and d_2 are allowed on the price of an article, obtain a formula for converting these into an equivalent single discount denoted by D.

Sug. Use $D=1-(1-d_1)(1-d_2)$.

Express your formula as a rule.

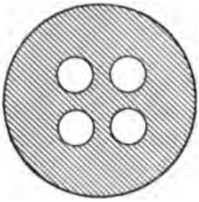
By use of the formula, convert two successive discounts of 40 % and 10 % into a single equivalent discount.

7. Denoting the number of square feet in the wall surface of a barn by F, the length of the barn by l, the width by w, the height at the eaves by e, and the height at the peak by p (all dimensions being in feet), find a formula for F in terms of the other letters.



If one gallon of paint will give two coats to 200 sq. ft. of surface, obtain a formula for G, the number of gallons required to paint the barn.

8. In the adjoining diagram, the radius of the large circle is R and of each of the small circles is r. Obtain a formula for A, the area of the part of the large circle left after the small circles have been cut out of it.



In the shortest way find A when $R=2.34$ in. and $r=.33$ in.

9. Obtain a formula for the weight of the pipe, in Ex. 5 if 1 cu. in. of iron weighs $\frac{1}{4}$ lb.

10. If d represents the size of a dose of medicine for an adult and D that for a child, and a the age of the child in years, then $D=\frac{ad}{a+12}$. Express this formula as a rule.

If 20 drops of a given medicine is a dose for an adult, what is the dose for a child 10 years old? For one 6 years old?

11. If the dimensions of a brick are a, b and c inches, what does $2ab+2ac+2bc$ represent? What does abc represent?

CHAPTER VIII

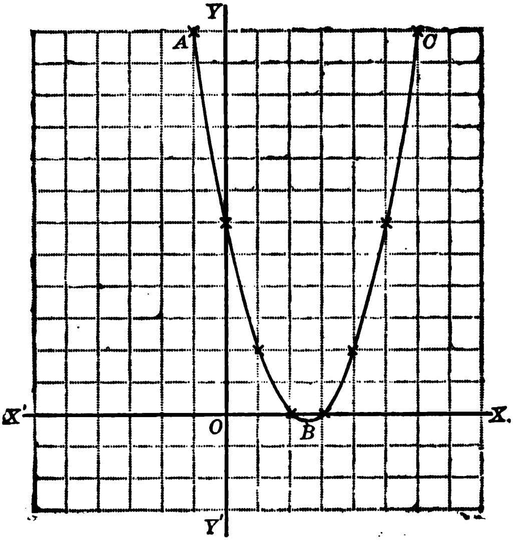
GRAPHS FOR QUADRATIC EQUATIONS

104. Graph of a Quadratic Equation of Two Unknown Quantities.

x	y
0	6
1	2
2	0
3	0
4	2
5	6
$2\frac{1}{2}$	$-\frac{1}{4}$
etc.	
-1	12
etc.	

Ex. 1. Construct the graph of $y=x^2-5x+6$.

The graph obtained is the curve ABC. A curve of this kind is called a *parabola*. The path of a projectile, as for instance, that of a baseball when thrown or batted (resistance of the air being neglected), is an arc of a parabola.

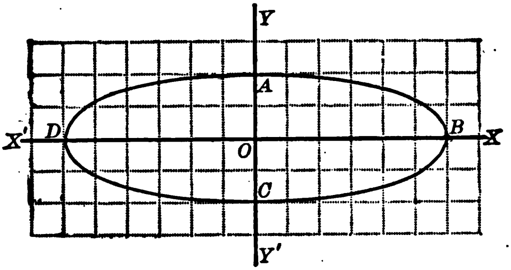


It will be noted that the above method of graphing is the same as that given in § 36 (p. 89), but that here it is sometimes advantageous to let x have fractional values, as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{10}$, $\frac{3}{2}$, etc. The observant pupil will also find methods of abbreviating the work in certain cases.

In general, the graph of a quadratic equation of two unknown quantities is a curved line, and, in particular, either a circle, parabola, ellipse, or hyperbola.

Ex. 2. Construct the graph of $x^2+9y^2=36$.

Before substituting numerical values for x, it saves labor to solve the given equation for y and to obtain $y=\pm\frac{1}{3}\sqrt{36-x^2}$.



x	y
0	± 2
1	$\pm 2-$
2	$\pm 1.9-$
3	$\pm 1.7+$
4	$\pm 1.5-$
5	$\pm 1.1+$
6	0
7	imag.

Since the only power occurring in the equation is x^2, and the square of a negative number is the same as the square of the corresponding positive number, the values of y for negative values of x are the same as for the corresponding positive values of x.

The species of curve obtained is called the *ellipse*. The practical value of the ellipse is illustrated by the fact that the earth's orbit about the sun and the orbits of all the other planets are ellipses. Also, in shape the earth, the sun, and planets are ellipsoids, that is, solids generated by the rotation of an ellipse about one of its axes, (as about AC in the above diagram).

Ex. 3. Construct the graph of $4x^2 - 9y^2 = 36$.

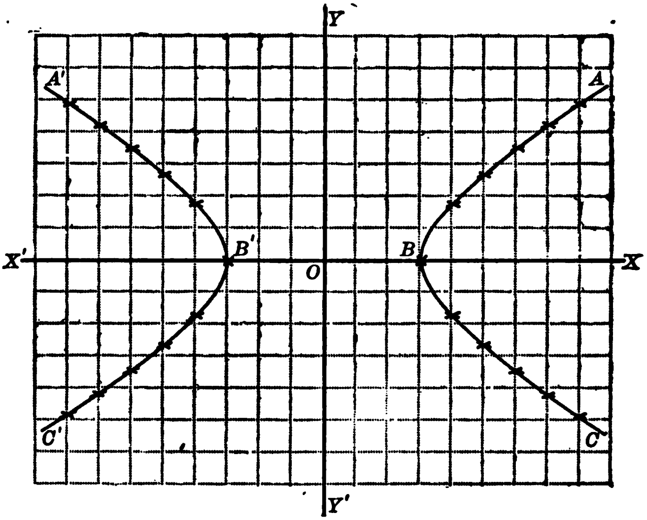
x	y
0	imag.
1	imag.
2	imag.
3	0
4	±1.7
5	±2.6
6	±3.4

etc.

Before substituting, solve and obtain

$$y = \pm\tfrac{2}{3}\sqrt{x^2 - 9}.$$

For negative values of x, the values of y are the same as for the corresponding positive values of x. Hence, the graph is a curve of two branches, ABC and $A'B'C'$, of the species known as the *hyperbola.*



105. Symmetry of Curves. What is a symmetrical plane figure as defined in geometry? How many axes of symmetry has a square? An equilateral triangle? A circle?

Note that in Ex. 2 on page 169 the ellipse is symmetrical to both the axis of x and the axis of y. The same is true of the hyperbola in Ex. 3 above.

How may labor be saved in constructing a graph that is symmetrical to one axis only? To both axes?

How can you tell from its equation whether a graph will be symmetrical to the axis of y? To the axis of x?

EXERCISE 92

Graph the following:

1. $y=x^2-2$.

2. $y=x^2+2x-3$.

3. $y=x^2-3x-4$.

4. $y=1.8x^2-2.3$.

5. $y=.7x^2$.

6. $y=20-x^2$.

7. $x^2+y^2=25$.

8. $x^2+y^2=\frac{9}{4}$.

9. $x^2+y^2=4.7$.

10. $y^2=9x-x^2$.

11. $16x^2+9y^2=144$.

12. $x^2+16y^2=64$.

13. $16x^2+y^2=64$.

14. $y^2-x^2=4$.

15. $x^2-y^2=4$.

16. $xy=3$.

17. $xy=-3$.

18. $x+xy=2$.

19. $(x-2)^2+y^2=9$.

20. $16y^2-x^2=-16$.

21. $y^2=9x+1$.

22. $x^2+xy+y^2=25$.

23. $x^2=9$.

24. $x^2-3x+2=0$.

Sug. In Ex. 24 show that whatever the value of y, x always equals 1 or 2. Hence the required graph is two straight lines parallel to the y-axis.

25. $y=18x^2$. Sug. Let each space on the y-axis $=20$.

26. $y^2=60x$.

27. $y=100x-16x^2$.

28. $x^2-xy+3y^2-4=0$.

29. $\frac{x+y}{x}=2x-1$.

30. Point out two of the above graphs which are symmetrical to the axis of x (but not to the axis of y).

31. Also three which are symmetrical to the axis of y (but not to the x-axis).

32. Also three which are symmetrical to both axes.

33. On one diagram, graph the results obtained by letting $a=1, 2, 4, 9$ in succession, in $x^2+y^2=a$.

34. Treat in like manner $xy=a$.

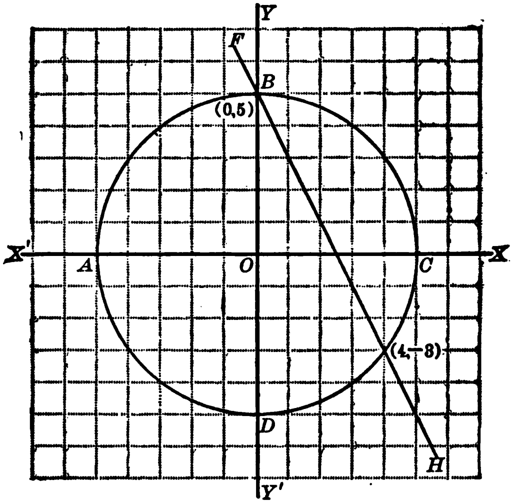
106. Graphic Solution of Simultaneous Quadratic Equations.

Ex. Solve graphically $\begin{cases} x^2+y^2=25. \\ y+2x-5=0. \end{cases}$

Constructing the graph of $x^2+y^2=25$, we obtain the circle *ABC* in the diagram below. Constructing the graph of $y+2x-5=0$, we obtain the straight line *FH*.

Measuring the coördinates of the points of intersection of the two graphs, we find the points to be (4, −3) and (0, 5).

These results may be verified by solving the two given simultaneous equations algebraically.



107. Special Cases: Imaginary Roots. Construct the graphs of $x^2+y^2=4$ and $x+y=5$. You will find that these two graphs do not intersect. Then solve the given equations in the ordinary algebraic way. You will find that the roots are imaginary. If you treat the equations $x^2+y^2=1$ and $4x^2+9y^2=36$ in the same way, you will obtain a similar result.

In general, *imaginary roots of simultaneous equations cor-*

respond to points of non-intersection of the graphs of the given equations.

Remember that in solving a pair of simultaneous equations, the number of values of x (and also of y) is equal to the sum of the degrees of the two equations. Hence, if two simultaneous equations are both of the second degree, their graphs should interesect in four points; and if their graphs are found to intersect in only two points, for instance, the other two points must correspond to imaginary roots.

The pupil may illustrate this by graphing and also solving algebraically $y^2=4x$ and $x^2+y^2=25$.

EXERCISE 93

Solve both algebraically and graphically:

1. $y^2=9x$.
$y-x=0$.

2. $y^2=9x$.
$y=3x$.

3. $y^2=2x$.
$x+y=4$.

4. $xy=4$.
$x+y=5$.

5. $xy=-3$.
$x+y=-2$.

6. $x^2+y^2=25$.
$xy=-12$.

7. $x^2+y^2=25$.
$3x+y-5=0$.

8. $x^2+y^2=25$.
$4y-3x=0$.

9. $x^2-4y=0$.
$3x^2+2y^2=14$.

10. $x^2+y^2-10y=0$.
$x=2y$.

11. $x^2+y^2=16$.
$x^2+9y^2=48$.

12. $4y^2-3x^2=24$.
$x^2+y^2=6$.

13. $3x^2+y^2=3$.
$y=x-2$.

14. $x-y-6=0$.
$y^2=6x-x^2$.

15. $y=x^2+2x-5$.
$3x=y+3$.

16. $x+\frac{1}{2}y=2$.
$x^2+y^2=9$.

17. $x^2+y^2=4$.
$3x-2y=12$.

18. $x^2=2y$.
$x^2+\frac{1}{4}y=9$.

Solve the following graphically, estimating the values of x and y to the nearest tenth:

19. $y^2=9x$.
$y-x=1$.

20. $x^2+y^2=16$.
$x+y=2$.

21. $xy=2$.
$x+y=5$.

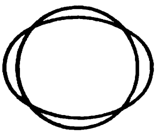
22. $x^2+y^2=11.$
$xy=4.$

23. $x-4y=4.$
$x^2+y^2=16.$

24. $x^2-4y=17.$
$3x=8+4y.$

25. $2x-3y=6.$
$3y+1=4x-4x^2.$

The largest number of points in which a circle can intersect an ellipse is illustrated by the adjoining diagram. Drawing a like diagram in each case, state the largest number of points in which the following pairs of lines can intersect:



26. An ellipse and an hyperbola. An ellipse and a parabola. A straight line and an ellipse.

27. A parabola and an ellipse. A parabola and an hyperbola. A straight line and a parabola.

28. The sum of the areas of two square pieces of tin is 25 sq. ft. A side of the larger piece is 1 ft. longer than a side of the smaller. Stating equations in two unknown quantities, find graphically a side of each square.

108. Graphic Solution of a Quadratic or Higher Equation of One Unknown Quantity. By substituting for y in the first equation, the pair of equations $y=x^2-5x+6$ and $y=0$ reduces to $x^2-5x+6=0$. Accordingly, the graphic solution of an equation like $x^2-5x+6=0$ is obtained by solving graphically $y=x^2-5x+6$ and $y=0$.

In other words, *the roots of a quadratic equation of one unknown quantity,* $ax^2+bx+c=0$, *are represented graphically by the abscissas of the points where the graph of* $y=ax^2+bx+c$ *meets the x-axis.*

Ex. Solve graphically $x^2-5x+6=0$.

The graph of $y=x^2-5x+6$ is the curved line ABC of the figure in § 104 (p. 168). This curve crosses the x-axis at the points (2, 0) and (3, 0). $\therefore x=2, 3.$ *Roots.* The same results are obtained by solving the equation $x^2-5x+6=0$ algebraically.

This method of solution applies also to a cubic equation or to an equation of one unknown quantity of any degree.

Thus to solve the equation $x^3-3x^2+5x-2=0$, graph the equation $y=x^3-3x^2+5x-2$. The abscissas of the points where this graph crosses the x-axis have the same value as the roots of the given equation $x^3-3x^2+5x-2=0$.

EXERCISE 94

Solve both graphically and algebraically:

1. $x^2-4=0$.
2. $x^2-3x-4=0$.
3. $x^2-4x+1=0$.
4. $4x^2+8x-5=0$.
5. $x^2-6x+9=0$.
6. $x^3-2x=0$.
7. $x^3-2x-1=0$.

SUG. Solve algebraically by the factor theorem.

8. $x^3-x^2-6x=0$.
9. $x^3-2x^2-2x+4=0$.

Solve the following graphically, estimating the roots to the nearest tenth:

10. $x^2-6x+4=0$.
11. $2x^2-4.5x+1.3=0$.
12. $x^2-1.7x-.5=0$.
13. $x^3+3x-5=0$.

EXERCISE 95

1. The population of a certain city for a series of years is given in the tabulation at the right. Make a graph of these figures and by extending the graph, make an estimate of the year when the population of the city will be one million.

YEAR	Population	YEAR	Population
1885	327,715	1905	546,203
1890	358,706	1910	616,287
1895	397,445	1915	693,203
1900	448,612	1920	782,916

2. In two districts the death rates per thousand for a period of years were as follows: Graph the facts and determine as accurately as you can in what year the death rates were the same in the city and country.

YEAR	1890	1893	1896	1899	1902	1905	1908	1911	1914	1917	1920
City	24.3	23.2	23.4	21.5	20.3	18.6	17.2	17.4	15.5	14.3	13.1
Country	16.8	17.1	16.5	17.9	18.7	19.5	19.8	20.3	21.2	20.8	19.2

3. The following table gives the wealth of the United States in billions of dollars and the population in millions:

YEAR	1850	1860	1870	1880	1890	1900	1910	1920
Wealth..........	7	16	30	44	65	89	165	
Population.......	23	31	39	50	63	76	92	
Wealth per capita								

Let the pupil fill in the per capita spaces in the tabulation, and then graph the three sets of numbers on one chart. What important fact can you learn from the chart?

4. The following tabulation gives the number of telephone calls in a residence section and also in a business section of a city for certain hours of the day:

FOR HOUR ENDING AT	7 A.M.	8 A.M.	9 A.M.	10 A.M.	11 A.M.	NOON	1 P.M.
Residence section	43	352	1763	3620	3819	3219	2720
Business section	62	187	423	2762	5267	4876	3818

FOR HOUR ENDING AT	2 P.M.	3 P.M.	4 P.M.	5 P.M.	6 P.M.	7 P.M.	8 P.M.
Residence section	2416	2304	2016	1920	1924	1870	1751
Business section	3746	4723	4210	3210	1690	420	127

Graph these two sets of numbers on the same chart.

Mention three facts of value which may be learned by comparing the two curves.

5. At noon on the third day from port, a steamer had 720 tons of coal in her bunkers and was using coal at the rate of 4 tons per hour. Show that the formula for the number of tons (T) in her hold with respect to the time named is $T=720-4\,h$.

Graph this formula, letting each space on the vertical or T-axis represent 60 tons. By means of this graph, determine the number of tons in her hold 30 hours after the time named. Also 20 hours before.

6. The formula for the area of a circle in terms of the diameter is $A=\frac{1}{4}\pi d^2$. Make a graph of this formula, letting each space on the vertical or A-axis represent 2 units of area.

By means of the graph, determine as accurately as you can the area of a circle whose diameter is $2\frac{1}{2}$ in. Also determine the diameter of the circle whose area is $5\frac{1}{2}$ sq. in. Is $8\frac{3}{4}$ sq. in.

7. The following are the data from which the temperature chart of a patient is to be made:

DAY OF DISEASE	1		2		3		4	
MORNING OR EVENING	M	E	M	E	M	E	M	E
Temperature		103°	102.2°	103.2°	102.5°	105°	102.5°	104.6°

DAY OF DISEASE	5		6		7	
MORNING OR EVENING	M	E	M	E	M	E
Temperature	102.6°	105.2°	102°	104.8°	103°	105°

[illegible] 95. 95.

[illegible] these facts [illegible] shown on the [illegible]

[illegible] has sizes of [illegible] are 1 [illegible] He [illegible] smallest [illegible] can be [illegible] and [illegible] Do [illegible] using the [illegible]

[illegible] edge of a piece [illegible] represent the [illegible] smallest size of pipe which he [illegible] following:

[illegible] a ½-in. pipe.

[illegible] ½-in. pipe.

CHAPTER IX

THEORY OF QUADRATIC EQUATIONS

109. Character of the Roots Inferred from the Coefficients. It is important to be able to infer at once from the nature of the coefficients of an equation whether the roots of the equation are equal or unequal, real or imaginary, positive or negative.

Any quadratic equation may be reduced to the form $ax^2+bx+c=0$, in which a is positive.

Solving $ax^2+bx+c=0$, and denoting the roots by r_1, r_2 (read, r sub-one, r sub-two), we obtain

$$r_1=\frac{-b+\sqrt{b^2-4\,ac}}{2\,a};\quad r_2=\frac{-b-\sqrt{b^2-4\,ac}}{2\,a}.$$

From these expressions we infer that

1. If $b^2-4\,ac$ *is* **positive,** *the roots are* **real** *and* **unequal.**
2. If $b^2-4\,ac$ *equals* **zero,** *the roots are* **real** *and* **equal.**
3. If $b^2-4\,ac$ *is* **negative** , *the roots are* **imaginary.**

The roots are rational if $b^2-4\,ac$ is a perfect square or zero.

Since the values of $b^2-4\,ac$ enable us to discriminate between different kinds of roots, this expression is termed the *discriminant* of $ax^2+bx+c=0$.

Ex. 1. Determine the character of the roots of the equation, $x^2-2\,x-1=0$.

We have $a=1,\quad b=-2,\quad c=-1.$

$$b^2-4\,ac=4+4=8.$$

Hence, the roots are *real* and *unequal.*

Ex. 2. Of $x^2-2x+1=0$.

Here $a=1,\quad b=-2,\quad c=1.$

$$b^2-4ac=4-4=0.$$

Hence, the roots are *real* and *equal*.

Ex. 3. Of $x^2-2x+2=0$.

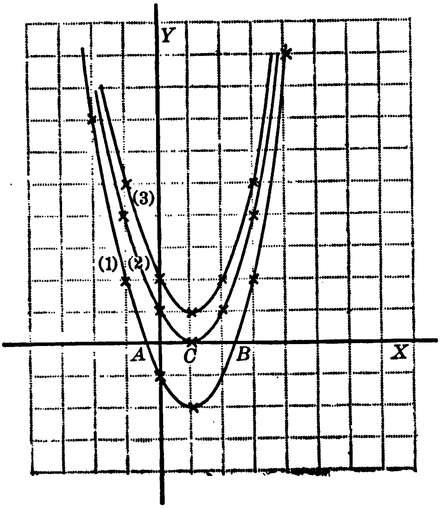
Here $a=1,\quad b=-2,\quad c=2.$

$$b^2-4ac=4-8=-4.$$

Hence, the roots are *imaginary*.

The results obtained in Exs. 1, 2, and 3 may be conveniently illustrated by means of graphs.

It is found that the graph of $y=x^2-2x-1$ is the curve (1) and crosses the x-axis at the two points A and B (corresponding to the two roots of $x^2-2x-1=0$).



The graph of $y=x^2-2x+1$ is the curve (2) and meets the x-axis at only one point (corresponding to the two equal roots of $x^2-2x+1=0$).

The graph of $y=x^2-2x+2$ is the curve (3) which does not meet the x-axis at all (which illustrates the fact that the roots of the equation $x^2-2x+2=0$ are imaginary).

110. Determining Coefficients so that the Roots shall Satisfy a Given Condition. It is often possible so to determine the coefficients of an equation that the roots shall satisfy a given condition.

Ex. Find the value of m which will give equal roots for the equation $(m-1)x^2+mx+2m-3=0$.

By § 109, 2, in order that the roots may be equal, $b^2-4ac=0$. In the given equation, $a=m-1$, $b=m$, $c=2m-3$.

$$\therefore m^2-4(m-1)(2m-3)=0.$$

$$m^2-8m^2+20m-12=0.$$

$$7m^2-20m=-12.$$

$$m=2,\ \tfrac{6}{7}.\quad \textit{Ans.}$$

Check. Substituting these values for m in the original equation,

$$x^2+2x+1=0,\qquad x^2-6x+9=0,$$

in each of which equations the roots are equal.

EXERCISE 96

Without solving, determine the character of the roots in

1. $x^2-5x+6=0$.
2. $3x^2-7x-2=0$.
3. $4x^2=4x-1$.
4. $3x^2+2x+1=0$.
5. $\dfrac{x-1}{3}=x^2-2$.
6. $x=-\dfrac{9x^2+4}{12}$.
7. $x=\frac{2}{3}(x^2+1)$.
8. $35x+18+12x^2=0$.
9. $\frac{1}{3}x^2=2x-3$.
10. $7x^2+1=5x$.

11. Determine by inspection the nature of the roots in the following:

(1) $x^2-4x+2=0$. (2) $x^2-4x+4=0$. (3) $x^2-4x+6=0$. Verify your results by use of graphs.

12. In $3x^2-2x+1=0$, determine the character of the roots by solving the equation. Now determine their character by the method of § 109. Compare the amount of work in the two processes.

13. What is the discriminant in Ex. 4? In Ex. 2?

14. What is the character of the roots in an equation in which the discriminant is -7.2? Is $\frac{4}{5}$? Is zero?

Determine the value of m for which the roots of each equation will be equal:

15. $2x^2-2x+m=0$.

16. $2x^2-mx+12\frac{1}{2}=0$.

17. $(m+1)x^2+mx=1$.

18. $(m+1)x^2+3m=12x$.

19. $(m+1)x^2+(m-1)x+m+1=0$.

20. $\dfrac{2x^2+3x+2}{7+3x-x^2}=\dfrac{1}{m}$.

21. In Ex. 15, for what values of m are the roots imaginary? Real and unequal?

22. Answer the same questions for Ex. 18.

23. Show that if one root of a quadratic equation is imaginary, the other root must be imaginary also.

24. State and prove a similar fact about irrational roots.

25. In $x^2-6x+c=0$, substitute (1) a value of c which will make the roots of the resulting equation equal; (2) a value which will make the roots imaginary; (3) a value which will make the roots real, unequal, and rational.

26. Make up a quadratic equation in which the roots are imaginary and the coefficient of x^2 is 2.

27. Also a quadratic equation in which the roots are equal and the coefficient of x is 12.

28. Also a quadratic equation in which the roots are real and unequal and the absolute term is 5.

111. Relation between Roots and Coefficients. In § 109, a method was obtained of inferring, from the coefficients of a quadratic equation, the *qualitative* nature of the roots. A more exact, or *quantitative*, relation between the roots and coefficients will now be obtained. This relation will enable us in any given equation to determine the sum or product (and often other functions) of the roots, without the labor of solving the equation.

By § 109 the roots of $ax^2+bx+c=0$ are

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad\text{and}\quad r_1=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Hence, $r_1+r_2=\dfrac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=\dfrac{-2b}{2a}=\dfrac{-b}{a}.$

that is, $r_1+r_2=\dfrac{-b}{a}.$

Also, $r_1r_2=\left(\dfrac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\dfrac{-b-\sqrt{b^2-4ac}}{2a}\right).$

$$=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{b^2-b^2+4ac}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Hence, if $ax^2+bx+c=0$ be reduced to the form, $x^2+\frac{b}{a}x+\frac{c}{a}=0$, in the latter form,

The sum of the roots equals the coefficient of x with the sign changed; and the product of the roots equals the absolute term.

Ex. 1. Without solving the equation, find the sum and also the product of the roots of $5(1-2x)=3x^2$.

The given equation reduces to $x^2+\frac{10}{3}x-\frac{5}{3}=0$.
Hence, sum of roots $=-\frac{10}{3}$, product of roots $=-\frac{5}{3}$. *Ans.*
The example may be checked by solving the given equation.

Ex. 2. Form the equation whose roots are $\dfrac{-1\pm\sqrt{-3}}{2}$.

The roots are $\dfrac{-1+\sqrt{-3}}{2},\ \dfrac{-1-\sqrt{-3}}{2}.$

Hence, sum of roots $=\dfrac{-1+\sqrt{-3}-1-\sqrt{-3}}{2}=\dfrac{-2}{2}=-1.$

Product of roots $=\dfrac{1-(-3)}{4}=\dfrac{1+3}{4}=1.$

Hence, $x^2+x+1=0$ is the required equation.
Check by solving the equation obtained.

EXERCISE 97

Find, by inspection, the sum and product of the roots in each of the following equations:

1. $x^2+3x+5=0.$

2. $x^2-x+7=0.$

3. $x^2-5x=10.$

4. $2x^2-6x-3=0.$

5. $x+1=\dfrac{x-5x^2}{x-1}.$

6. $a^2x^2-ax+2=0.$

7. $5x-4x^2=1.$

8. $3-7x=11x^2.$

9. $\dfrac{x}{a}=\dfrac{x-a}{4x+1}.$

10. $1-2cx-2ax^2=3c.$

Form the equations whose roots are

11. 2, 3.

12. 3, -2.

13. -1, -5.

14. 5, .04.

15. -1, $-\frac{2}{3}$.

16. $\frac{2}{3}$, $-\frac{2}{3}$.

17. .08, $-.2$.

18. ab, $-a$.

19. $\dfrac{a}{b}$, $-\dfrac{b}{a}$.

20. $1+\sqrt{2}, 1-\sqrt{2}.$

21. $-3\pm\sqrt{3}.$

22. $\dfrac{2\pm\sqrt{2}}{2}.$

23. $\dfrac{1\pm\sqrt{-1}}{2}.$

24. $\dfrac{-2\pm\sqrt{-2}}{2}.$

25. $\frac{1}{2}a\pm c\sqrt{-b}.$

26. If one root of the equation $x^2-\frac{13}{5}x-\frac{6}{5}=0$ is 3, find the other root in two different ways.

27. If one root of $x^2-2x-1=0$ is $1-\sqrt{2}$, find the other root in three different ways.

By use of the principle proved in § 111, show whether.

28. 1 and $-\frac{7}{3}$ are roots of $3x^2+4x-7=0.$

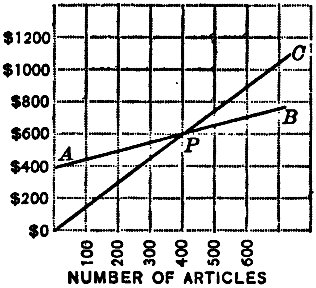
29. $\frac{1}{4}$ and $-\frac{3}{2}$ are the roots of $\dfrac{1}{2}=\dfrac{7x}{6}+x^2.$

30. .34 and $-.5$ are the roots of $x^2+.16x-.17=0.$

31. In the equation $x^2=5x-c$, find the value of c, if the difference between the roots is 11.

112. Net Profit Graph.

Ex. The cost of preparing the machinery to manufacture a certain article is $400, after which the cost of manufacturing each article is 50 cents. If each article sells for $1.50, how many articles must be manufactured and sold before the process becomes profitable? Solve graphically.



Let each space on the vertical axis represent $200 and each space on the horizontal axis represent 100 articles. The diagram shows that 400 articles must be made and sold before a net profit begins.

Let the pupil check the solution by use of formulas.

EXERCISE 98

1. The cost of preparing the machinery to manufacture a certain article is $300, after which the cost of manufacturing each article is 25 cents. If each article sells for one dollar, show graphically how many articles must be made and sold before the process becomes profitable.

2. The cost of preparing the machinery to manufacture a certain article is m dollars, after which the cost of manufacturing each article is b dollars. The article sells for s dollars. Obtain a formula for n, the number of articles that must be made and sold before there is a net profit.

3. The tabulation on page 186 gives facts concerning the circulation of a certain newspaper, the number of advertisements printed and the population of the city where the paper is published. All these figures are in thousands.

After charting these facts, from an examination of the graphs, answer the following questions: Which grew more rapidly, the population of the city or the circulation of the

Year	1890	1895	1900	1905	1910	1915	1920
Circulation	32	36	42	47	58	69	83
Number of ads.	51	63	87	108	141	192	263
Population	112	121	136	148	153	162	173

newspaper? The circulation of the newspaper or the number of advertisements printed in it?

4. The following tabulation shows the population of the United States in millions for the years named; also the foreign commerce of the United States in millions of dollars. Let the pupil calculate and fill in the per capita facts.

Year	1800	1810	1820	1830	1840	1850	1860	1870	1880	1890	1900	1910	1920
Population	5	7	10	13	17	23	31	39	50	63	76	92	
Commerce				134	222	318	687	829	1504	1680	2307	3429	
Per capita													

Graph the facts on one diagram and state three facts which you can learn from an examination of your chart.

5. Construct a graph of the formula $C=\frac{5}{9}(F-32)$, letting each space on the vertical (or Centigrade) scale represent 5 degrees and each space on the horizontal (or Fahrenheit) scale represent 10 degrees.

By use of this graph, determine the temperature on the other scale which corresponds to 0° F. To 32° F. To 100° F. To −5° F. To 10° C. To 60° C. To −12° C.

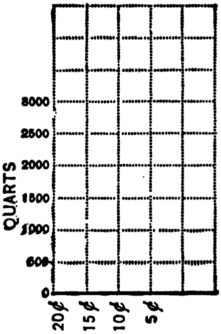
6. By use of the chart made in Ex. 6, also find that temperature which is represented by the same number of degrees on both the Fahrenheit and Centigrade scales.

Sug. On the diagram construct the graph of $C=F$ and observe where it intersects the graph of $C=\frac{5}{9}(F-32)$.

Check your answer algebraically by solving $C = \frac{5}{9}(F - 32)$ and $C = F$ as simultaneous equations.

7. A wholesale fruit dealer found during a certain season that when the price of strawberries was 20 cents per quart he sold 100 qt. per day; when the price was 15 cents he sold 500 qt. daily; when the price was 10 cents he sold 1000 qt.; when price was 5 cents he sold 3000 qt. Construct a graph for these facts. (Let each space on the horizontal axis represent 5 cents and each space on the vertical axis represent 500 qt.)

A graph of facts like these is called a *demand curve.*



For the next year he expected that the demand would be 20 per cent greater. On the same diagram with the first curve construct the expected demand curve for the next year.

Construct graphs for the data in Exs. 8, 9, 10, in each case using that one of the three forms; viz., curves, bars, or the circle, which you think is most suitable.

8. The lengths in miles of certain representative rivers in the United States are as follows: Hudson, 280; Ohio, 950; Colorado, 1360; Mississippi, 3160.

9. The corn crop of the United States in millions of bushels, and the population in millions, for the years specified were as follows:

YEAR	1840	1850	1860	1870	1880	1890	1900	1910	1920
Population	17	23	31	39	50	63	76	92	
Corn crop	377	592	839	1094	1717	1689	2151	2886	
Per capita									

10. A man's total income for a certain year was \$5 The sources from which it was derived were as follo Salary, \$3000; house rents, \$700; dividends from stock, \$1100; literary work, \$400.

EXERCISE 99

REVIEW

1. Factor:

(1) $ax+4a-3x-12.$ (3) $a^3+3a^2-4.$

(2) $4-x(4+3x).$ (4) $(a^2-b^2-c^2)^2-4b^2c^2.$

Simplify: **2.** $\frac{1}{(x+2)(x-3)}-\frac{1}{(3-x)(x-2)}.$

3. $\left(1-\frac{a}{a-m}\right)+\left(1-\frac{a}{a+m}\right).$

4. Solve for x and check: $\frac{1}{ab-ax}+\frac{1}{bc-bx}-\frac{1}{ac-ax}=0.$

Solve for x and y and check:

5. $\frac{1}{10}(3x+4y-3)-\frac{1}{15}(2x+7-y)=4\frac{2}{5}+\frac{1}{5}(y-8).$
$\frac{1}{12}(9y+5x-8)-\frac{1}{4}(x+y)-\frac{1}{11}(7x+6)=0.$

6. $(a+b)x-(a-b)y=3ab.\quad(a-b)x-ay-by=ab.$

7. B worked 3 days on a job after which A, working alone, completed it in 6 days. Working together they could have done it in 4 days. How long would it take each one separately?

8. If one tank contains water at 210° and another at 110°, how many gallons must be taken from each tank to make 120 gallons at 150°?

9. Extract the square root of $9x^{-3}-30x^{-\frac{5}{2}}y+13x^{-2}y^2+20x^{-\frac{3}{2}}y^3+4x^{-1}y^4.$

Simplify:

10. $\left(\frac{x^{-1}y^2}{a^2b^{-4}}\right)^{-2}+\left(\frac{xy^{-1}}{a^{-2}b^2}\right)^3.$ **11.** $\frac{(a^n)^{n-1}}{a^n\cdot a^{n-1}}.$

12. $3^{-2}-5\times4^0+8^{-\frac{2}{3}}+1^{-\frac{2}{3}}-3(a+b)^0.$

13. Multiply $3\sqrt{2x}-5\sqrt{x-1}$ by $3\sqrt{2x}+5\sqrt{x-1}.$

14. Evaluate $\frac{5\sqrt{7}-1}{\sqrt{7}+2}$ to three decimal places in the shortest way.

15. Solve $\sqrt{9+2x}-5\div\sqrt{9+2x}=\sqrt{2x}$, and check.

Solve for x:

16. $\sqrt{x+5}+\sqrt{2x+8}-\sqrt{7x+21}=0$.

17. $3x^{\frac{1}{2}}=8x^{\frac{1}{4}}-4$.

18. $abx^2+(a^2+b^2)x+ab=0$.

19. Find three consecutive numbers such that their sum equals the product of the last two of the numbers.

20. One baseball nine has won 6 games out of 15 and another 8 out of 13. How many straight games must the first win from the second in order that their averages shall be equal?

Solve and check:

21. $x^2+3xy=7$.
$y^2+xy=6$.

22. $x^3+y^3=218a^3$.
$x^2-xy+y^2=109a^2$.

23. Find two numbers whose difference is 6 and such that the sum multiplied by the less equals 176.

24. The sum of the terms of a given fraction is 14. If the numerator is diminished by 2 and the denominator by 3, the product of the resulting fraction by the original fraction is $\frac{5}{18}$. Find the fraction.

25. Given that -3.2 is one root of $x^2+7.7x+14.4=0$, calculate the other root (1) from the coefficient of x; (2) from the absolute term.

26. Without solving or substituting, determine whether 2 and 5 are the roots of $2x^2-7x-15=0$.

27. Find the value of k which makes the roots equal in $x^2+25=9x+kx$.

28. Solve $L=l+\frac{8d^2}{3l}$ for l.

29. Solve $L=\frac{wh}{d(w+p)}$ for w.

30. To the nearest hundredth evaluate $2.42\left(1+\frac{t}{T}\right)\sqrt{h}$, when $t=10$, $T=15$, and $h=7$.

31. A box (without lid) is made of boards t inches thick. If the outside dimensions of the box are l, w, h inches, obtain a formula for the number of board feet, B, used in making the box.

32. Plot $xy=4$ and $2x-3y=5$ on the same axes and from the figure estimate the values of x and y.

33. Eliminate c between $R=a\left(\frac{1}{c}+d\right)$ and $2a=3c$.

34. A box car is l feet long and w feet wide. It is filled d feet deep with grain. If $1\frac{1}{4}$ cu. ft. of the grain weighs 60 lb., obtain a formula for w, the weight of the grain in pounds.

Graph the numerical facts given in Exs. 35, 36, 37, in each case selecting the most appropriate form of graph.

35. In a recent year the per cents of persons in the United States engaged in various gainful occupations were as follows:

Agriculture,	35 %	Domestic service,	10 %
Manufacturing,	26 %	Professional and clerical work,	9 %
Trade and transportation,	16 %	Other occupations,	4 %

36. In the year 1860 the average production of cereals per farm worker in the United States was 266 bushels; in the year 1919 it was 418 bushels. Can you give any reason for this difference?

37. The following tabulation gives the population of the United States in millions, and the number of thousands of miles of railroad in the years specified:

YEAR	1840	1850	1860	1870	1880	1890	1900	1910	1920
Population	17	23	31	39	50	63	76	92	
R. R.m'l'ge	3	9	31	53	93	167	199	250	
No. mi. per 1000 pop.									

38. If water is running through a pipe d inches in diameter at the rate of f feet per minute, obtain a formula for the number of minutes (M) that will be required to cover an acre of ground 1 in. deep.

39. A man has an income of i dollars. All of this above d dollars is subject to a tax of a per cent, and all above e dollars is subject to an additional tax of b per cent. Obtain a formula for T, his total income tax.

CHAPTER X

THE PROGRESSIONS

113. A Series is a succession of terms formed according to some law, as

$$1, 4, 9, 16, 25, \ldots$$
$$1 - x + x^2 - x^3 + x^4 -, \ldots$$
$$2, 4, 8, 16, 32, \ldots$$

114. Utility in an Algebraic Treatment of Series.

Ex. If a car going down an inclined plane travels in successive seconds, 2 ft., 6 ft., 10 ft., 14 ft., etc., how far will it go in 30 seconds?

The direct method of solution would be to set down the 30 numbers involved and add them. But by investigating the laws of the series involved and expressing these as formulas, it will be found (see § 117) that this long addition can be converted into two short multiplications and much labor can thus be saved.

The algebraic study of the laws of series will enable us to save labor in various ways, and to obtain other important results.

ARITHMETICAL PROGRESSION

115. An Arithmetical Progression is a series each term of which is formed by adding a constant quantity, called the *difference*, to the preceding term.

Thus, 1, 4, 7, 10, 13, . . . is an arithmetical progression (denoted by A. P.) in which the difference is 3.

Given an arithmetical progression, to determine the difference: *from any term (except the first) subtract the preceding term.*

Thus, in the A.P., $\frac{3}{2},\ -\frac{3}{4},\ -3,$
the difference $=-\frac{3}{4}-\frac{3}{2}=-\frac{9}{4}.$

116. Quantities and Symbols. In an A. P. we are concerned with five quantities:

1. The *first term*, denoted by a.
2. The *common difference*, denoted by d
3. The *last term*, denoted by l.
4. The *number of terms*, denoted by n.
5. The *sum of the terms*, denoted by s.

117. Two Fundamental Formulas. Since in an A. P. each term is formed by adding the common difference, d, to the preceding term, the general form of an A. P. is

$$a,\quad a+d,\quad a+2d,\quad a+3d,\ \ldots$$

Hence, the coefficient of d in each term is one less than the number of the term.

Thus, the 7th term is $a+6d$,
12th term is $a+11d$,
nth term is $a+(n-1)d$.

Hence, $$l=a+(n-1)d. \qquad (1)$$

Also,

$$s=a+(a+d)+(a+2d)+\ldots+(l-d)+l. \qquad (2)$$

Writing the terms of this series in reverse order,

$$s=l+(l-d)+(l-2d)+\ldots+(a+d)+a. \qquad (3)$$

Adding (2) and (3),

$$\begin{aligned}2s&=(a+l)+(a+l)+(a+l)+\ldots+(a+l)+(a+l)\\&=n(a+l).\end{aligned}$$

$$\therefore\ s=\frac{n}{2}(a+l). \qquad (4)$$

If we substitute for l in (4) from (1),

$$s=\frac{n}{2}[2\,a+(n-1)d]. \quad \ldots\ldots\ldots \quad (5)$$

Hence, combining results, we have the two fundamental formulas for l and s,

$$\text{I.}\quad l=a+(n-1)d.$$

$$\text{II.}\quad s=\frac{n}{2}(a+l).$$

$$s=\frac{n}{2}[2\,a+(n-1)d].$$

Thus, Formula I substitutes a multiplication for successive additions of the common difference; and Formula II substitutes a multiplication for the addition of the successive terms.

Ex. 1. Find the 12th term and the sum of 12 terms of the A. P., 5, 3, 1, −1, −3, ...

In this series $a=5,\ d=-2,\ n=12.$
From I, $l=5+(12-1)(-2)=5-22=-17.$
From II, $s=\frac{12}{2}(5-17)=-72.$ *Sum.*

Ex. 2. Find the sum of n terms of the A. P.,

$$\frac{a+b}{2},\ \frac{a-b}{2},\ \frac{a-3\,b}{2},\ \ldots$$

Here $a=\dfrac{a+b}{2},\ d=-b,\ n=n.$

Substituting in the fundamental formula, $s=\frac{n}{2}[2\,a+(n-1)d]$,

$$s=\frac{n}{2}[a+b+(n-1)(-b)]$$

$$=\frac{n}{2}[a+(2-n)b]. \quad \textit{Sum.}$$

EXERCISE 100

1. Give the value of d in Exs. 2–15.

2. Find the 8th term in the series 3, 7, 11, . . .

3. Find the 9th term and the sum of 9 terms in 7, 3, -1, . . .

4. Find the 20th and 28th terms in 5, $\frac{13}{3}$, $\frac{11}{3}$, . . .

5. Find the 16th and 25th terms in $-13\frac{1}{2}$, -9, $-4\frac{1}{2}$, . . .

6. Find the 10th term and sum of 10 terms of $1.12, $1.06, $1, $.94, . . .

Find the sum of the series:

7. 3, 8, 13, . . . to 8 terms.

8. 3, -3, -9, . . . to 9 terms.

9. $-\frac{1}{3}$, $\frac{1}{2}$, $\frac{4}{3}$, . . . to 38 terms.

10. $-\frac{3}{4}$, $-\frac{5}{6}$, $-\frac{11}{12}$, . . . to 55 terms.

11. $5\sqrt{2}-2\sqrt{3}$, $4\sqrt{2}-3\sqrt{3}$, . . . to 11 terms.

12. $3a-\frac{1}{a}$, $2a$, $a+\frac{1}{a}$, . . . to 12 terms.

13. 6, 3, 0, -3, -6, . . . to p terms.

14. 5, 3, 1, -1, . . . to n terms.

15. $2x-y$, $x+y$, $3y$, . . . to r terms.

16. Find the sum of the first 30 odd numbers by writing them down and adding them. Now find their sum by use of the formula. Compare the amount of work in the two processes.

17. How many strokes does a clock make in striking each hour of the day?

18. If a man saves $100 in his 20th year, $150 the next year, and $200 the next, and so on through his 50th year, how much will he save in all?

19. A body rolling down an inclined plane goes 6 ft. in the first second, three times as far in the next second, 5 times as far in the third second, and so on. How far will it go in 10 seconds?

118. Given any Three of the Five Quantities, a, d, l, n, s, to Find the Other Two.

The method in general is as follows:

If the three known quantities are found in one of the formulas of § 117, *substitute the three given values in the formula and solve the resulting equation.*

The remaining unknown quantity may then be found by use of one of the other formulas of § 117.

If the three given quantities do not occur in one of the formulas of § 117, *substitute in two of these formulas and solve by elimination.*

Ex. 1. Given $l=13$, $s=49$, $n=7$, find a and d.

Since the letters l, s, n, and a all occur in the formula $s=\frac{n}{2}(a+l)$, substitute the values of l, s, and n in this formula.

Hence, $49=\frac{7}{2}(a+13)$,

$98=7a+91. \quad \therefore a=1.$ *Ans.*

From $l=a+(n-1)d$,

$13=1+(7-1)d. \quad \therefore d=2.$ *Ans.*

Ex. 2. Given $d=2$, $l=21$, $s=121$, find a and n.

Since d, l, and s do not occur in one formula, we substitute for d, l, s in Formulas I and II.

Hence,
$$21=a+(n-1)2, \qquad (1)$$
$$121=\frac{n(n+21)}{2}. \qquad (2)$$
$$\therefore a+2n=23, \qquad (3)$$
$$an+21n=242. \qquad (4)$$

Substitute for a in (4) from (3),

$$n(23-2n)+21n=242.$$

Whence, $n=11.$
Hence, from (3), $a=1.$ } *Ans.*

EXERCISE 101

Find the first term and the sum of the series when

1. $d=3$, $l=40$, $n=13$. **2.** $d=\frac{3}{4}$, $l=18\frac{1}{3}$, $n=33$.

Find the first term and the common difference, when

3. $s=275$, $l=45$, $n=11$. **4.** $s=4$, $l=-10$, $n=8$.

5. $s=-\frac{47}{6}$, $l=-4$, $n=47$.

6. For an article bought on the installment plan, the interest on unpaid balances is the following series: \$.08, \$.16, \$.24, . . . , up to \$1.92. Find their sum.

Find n and d when

7. $a=-5\,l$, $l=15$, $s=105$. **8.** $a=\frac{7}{12}$, $l=-\frac{5}{6}$, $s=-2\frac{1}{4}$.

Find a and n when

9. $s=10$, $d=3$, $l=8$. **10.** $s=10$, $d=-3$, $l=-4$.

11. $l=-8$, $d=-3$, $s=-3$.

12. $l=-\frac{4}{3}$, $d=-\frac{1}{15}$, $s=-\frac{40}{3}$.

How many consecutive terms must be taken in the series

13. 1, $1\frac{1}{2}$, 2, . . . to make the sum 45?

14. $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, . . . to make the sum -1?

15. $\frac{3}{2}$, $\frac{5}{4}$, 1, . . . to make the sum 4.5?

16. A body rolling down an inclined plane goes 6 ft. the first second, 18 ft. the next second, 30 ft. the third second, and so on. In how many seconds will it have traveled 486 ft.?

119. Arithmetical Means.

Ex. Insert 9 arithmetical means between 1 and 5.

We have given $a=1$, $l=5$, $n=11$. Hence, we find $d=\frac{2}{5}$.

The required means are therefore $1\frac{2}{5}$, $1\frac{4}{5}$, $2\frac{1}{5}$, . . . *Ans.*

In case only a single arithmētical mean is to be inserted between two quantities, a and b, this one mean is found most readily by use of the formula $\frac{a+b}{2}$. For if x denotes the required mean, the A. P. is a, x, b.

Hence, $$x-a=b-x.$$

$$2x=a+b.$$

$$x=\frac{a+b}{2}.$$

EXERCISE 102

Insert

1. Four arithmetical means between 7 and -3.
2. Seven arithmetical means between 4 and 6.
3. Fifteen arithmetical means between $-4\frac{1}{3}$ and 9.
4. The arithmetical mean between $2\frac{1}{5}$ and $-5\frac{2}{3}$.
5. The arithmetical mean between $p+1$ and $p-1$.
6. Find the A. M. between $\frac{p}{q}$ and $\frac{q}{p}$. $\frac{1}{r+s}$ and $\frac{1}{r-s}$.

7. If the height of Bunker Hill Monument is 221 ft., of the Washington Monument 555 ft., and the length of the Olympic 882 ft., by how much does the middle one of these numbers differ from the arithmetical mean between the other two?

8. Rome was founded 753 B. C. and fell A. D. 476. How far is the latter number from being an arithmetical mean between the former and the number of the year in which Columbus discovered America?

9. Arithmetical means are to be inserted between 1 and 21 in such a way that their sum shall be 132. Find the first two of these means.

10. Show that if twice one number equals the sum of two other numbers, the three numbers may be arranged as an A. P.

120. Miscellaneous Examples.

Ex. 1. The 7th term of an A. P. is 5, and the 14th term is -9. Find the first term.

By the use of Formula I (§ 117),

the 7th term is $a+6d$, and the 14th term is $a+13d$.

$$\therefore a+6d=5. \quad \ldots \quad (1)$$

$$a+13d=-9. \quad \ldots \quad (2)$$

Subtracting (1) from (2), $7d=-14$.

$$d=-2.$$

Substitute for d in (1), $a-12=5$.

$a=17$. *Ans.*

Ex. 2. The sum of five numbers in A. P. is 15, and the sum of the 1st and 4th numbers is 9. Find the numbers.

Denote the numbers by

$$x-2y, \quad x-y, \quad x, \quad x+y, \quad x+2y.$$

Add, $5x=15. \quad \ldots \quad (1)$

Also $(x-2y)+(x+y)=9.$

$$2x-y=9. \quad \ldots \quad (2)$$

From (1) $x=3$; hence, from (2) $y=-3$.

Hence, the numbers are 9, 6, 3, 0, -3. *Ans.*

Similarly, in dealing with four unknown quantities in A. P., we may denote them by

$$x-3y, \quad x-y, \quad x+y, \quad x+3y.$$

EXERCISE 103

Find the first two terms of the series wherein

1. The 4th term is 11 and the 10th is 23.
2. The 6th term is -3 and the 12th is -12.
3. The 5th term is $c-3b$ and the 11th is $3b-5c$.
4. Find the sum of the first n odd numbers. State the result obtained as a rule in general language.

5. Set down the first 20 odd numbers and find their sum by addition. Now find their sum by the formula result obtained in the preceding example. Compare the amount of work in the two processes.

6. Find the sum of the first n numbers divisible by 7.

7. Which term in the series $1\frac{1}{6}$, $1\frac{1}{3}$, $1\frac{1}{2}$, . . . is 18?

8. The first term of an arithmetical progression is 8; the 3d term is to the 7th as the 8th is to the 10th. Find the series.

9. Find four numbers in A. P., such that the sum of the first two is 1, and the sum of the last two is -19.

10. Find four numbers in A. P. whose sum is 16 and whose product is 105.

11. A man travels $2\frac{1}{3}$ mi. the first day, $2\frac{2}{3}$ the second, 3 the third, and so on; at the end of his journey he finds that if he had traveled $6\frac{1}{6}$ mi. every day he would have required the same time. How many days was he walking?

12. The sum of 10 numbers in an A. P. is 145, and the sum of the fourth and ninth terms is 5 times the third term. Find the series.

13. If the 11th term is 7 and the 21st term is $8\frac{2}{3}$, find the 41st term of the same A. P.

14. In an A. P. of 21 terms the sum of the last three terms is 23, and the sum of the middle three is 5. Find the series.

15. Required five numbers in A. P., such that the sum of the first, third, and fourth terms shall be 8, and the product of the second and fifth shall be -54.

16. The sum of the first 8 terms of an A. P. is 64, and the sum of the first 18 terms of the same A. P. is 324. Find the series.

17. The 5th term of an A. P. is 13, and the 13th term is 37. Find the sum of 21 terms of the A. P.

18. The 14th term of an A. P. is 38; the 90th term is 152; and the last term is 218. Find the number of terms.

19. How many numbers of two figures are there divisible by 3? By 7? How many numbers of three figures are divisible by 6? By 9?

20. How many numbers of four figures are there divisible by 11? Find the sum of all the numbers of three figures divisible by 7.

21. If a car starts at the top of a hill and runs down an inclined track 2 ft. the first second, 6 ft. the next second, 10 ft. the next, etc., and reaches the bottom in 12 seconds, how long is the track?

22. If a body falls $16\frac{1}{12}$ ft. in the first second; three times this distance in the next second; five times in the third, and so on, how far will it fall in the 30th second? How far will it have fallen during the 30 seconds? In how many seconds will it have fallen $6433\frac{1}{3}$ ft.?

23. If a, b, c, d, are in A. P., prove: (1) that $a+d=b+c$; (2) that ak, bk, ck, dk, are also in A. P.; and (3) that $a+k$, $b+k$, $c+k$, $d+k$ are in A. P. State this problem without the use of the symbols, a, b, c, d, k; that is, in general language.

GEOMETRICAL PROGRESSION

121. A Geometrical Progression is a series each term of which is formed by multiplying the preceding term by a constant quantity called the *ratio.*

Thus, 1, 3, 9, 27, 81, . . . is a geometrical progression (or G. P.) in which the ratio is 3.

Given a geometrical progression, to determine the ratio: *divide any term by the preceding term.*

Thus, in the G. P., $-3,\ \frac{3}{2},\ -\frac{3}{4},\ \ldots$

the ratio $=\dfrac{\frac{3}{2}}{-3}$, or $-\dfrac{1}{2}$.

122. Quantities and Symbols. The symbols a, l, n, s have the same meaning as in A. P. Besides these, r is used to denote the ratio.

123. Two Fundamental Formulas. Since in a G. P. each term is formed by multiplying the preceding term by the common ratio, r, the general form of a G. P. is

$$a,\quad ar,\quad ar^2,\quad ar^3,\quad ar^4,\ \ldots$$

Hence, the exponent of r in each term is one less than the number of the term.

Thus, the 10th term is ar^9,

15th term is ar^{14},

nth term, or $l=ar^{n-1}$. . . . (1)

In deriving a formula for the sum, we know, also,

$$s=a+ar+ar^2+\ldots+ar^{n-1}. \qquad (2)$$

Multiply (2) by r,

$$rs=ar+ar^2+ar^3+\ldots+ar^{n-1}+ar^n. \qquad (3)$$

Subtract (2) from (3),

$$rs-s=ar^n-a$$

$$\therefore\ s=\frac{ar^n-a}{r-1}. \qquad (4)$$

Multiply (1) by r, $rl=ar^n$.

Substitute rl for ar^n in (4),

$$s=\frac{rl-a}{r-1}. \qquad (5)$$

Hence, collecting the results obtained in (1), (4), (5) we have the two fundamental formulas for l and s:

$$\text{I.}\quad l=ar^{n-1}.$$

$$\text{II.}\quad s=\frac{ar^n-a}{r-1}.$$

$$s=\frac{rl-a}{r-1}.$$

Ex. 1. Find the 8th term and sum of 8 terms of the G. P., 1, 3, 9, 27, . . .

In this case $a=1,\ r=3,\ n=8.$

From I, $l=1\times3^7=2187.$

From II, $s=\dfrac{3\times2187-1}{3-1}=3280.$ *Ans.*

Ex. 2. Find the 10th term and the sum of 10 terms of the G. P., 4, -2, 1, $-\frac{1}{2}$, . . .

Here $a=4,\ r=-\frac{1}{2},\ n=10.$

Hence, $l=4(-\frac{1}{2})^9=-\frac{4}{512}=-\frac{1}{128}.$

$$s=\frac{(-\frac{1}{2})(-\frac{1}{128})-4}{-\frac{1}{2}-1}=\frac{341}{128}.\quad \textit{Sum.}$$

EXERCISE 104

Give the value of r in Exs. 2–15.

1. Find the 6th term in the series 2, 6, 18, ...
2. Find the 7th term in 3, 6, 12, . . .
3. Find the 6th and the sum of 6 terms in 45, -15, 5, ...
4. Find the 5th and the sum of 5 terms in 81, -54, . . .
5. Find the 4th and the sum of 4 terms of 1.05, 1.05^2, . . .
6. Find the 9th term in the series 2, $2\sqrt{2}$, 4, . . .
7. 15th term of $\frac{1}{b}$, $\frac{1}{b^2}$, $\frac{1}{b^3}$, . . .
8. nth term of p, $\frac{p}{q}$, $\frac{p}{q^2}$, $\frac{p}{q^3}$, . . .

Find the sum of the series

9. $3, -6, 12, \ldots$ to 6 terms.

10. $27, -18, 12, \ldots$ to 7 terms.

11. $\frac{2}{3}, -\frac{1}{6}, \frac{1}{24}, \ldots$ to 8 terms.

12. $\frac{1}{\sqrt{3}}, 1, \sqrt{3}, \ldots$ to 8 terms.

13. $\sqrt{2}-1, 1, \sqrt{2}+1, \ldots$ to 6 terms.

14. $1, 2, 4, 8, \ldots$ to n terms.

15. The following is a series of specific gravities: cork, .25; oak wood, .75; aluminum, 2.5; iron, 7.5; platinum, 21.5. By how much does each term of this series differ from the corresponding term in a G. P. whose first term is .25 and whose ratio is 3?

(What is meant by specific gravity?)

16. If the average age of parents is taken as 30 years, find the total number of a person's ancestors in a period of 600 years.

17. The population of the United States in the year 1900 was 76,300,000. If this should increase 50 % every 25 years, what would the population be in the year 2000?

18. If a man saves $300 each year for 10 years, what is the amount of his savings in 5 years at compound interest at 5 per cent? In 10 years?

19. A ship was built at a cost of $70,000. Her owners at the end of each year deducted 10 % from her value as estimated at the beginning of the year. What is her estimated value at the end of 10 years?

20. A grain of wheat, when planted, produced a stalk on which were 30 other grains. The next year each of these 30 grains was planted and produced similar stalks. If this process were continued, at the end of 10 years h many bushels would be produced in the last crop if 1 quart contains 2000 grains?

124. Given Three of the Five Quantities a, l, n, s, r, to Determine the Other Two.

Use the same general method as that given in § 118 (p. 195), *for* A. P.

Ex. 1. Given $a=-2$, $n=7$, $l=-128$; find r, s.

From I, $-128=-2r^6$.

Hence, $r^6=64$, $r=\pm 2$.

From II, if $r=+2$, $s=\dfrac{2(-128)-(-2)}{2-1}=-256+2=-254$.

If $r=-2$, $s=\dfrac{(-2)(-128)-(-2)}{-2-1}=\dfrac{256+2}{-3}=-86$.

Hence, there are two sets of answers; viz.:

$$\left.\begin{array}{ll} r=2, & s=-254. \\ r=-2, & s=-86. \end{array}\right\} \quad \textit{Ans.}$$

Ex. 2. Give $a=\frac{3}{4}$, $r=-\frac{1}{3}$, $s=\frac{91}{162}$; find l, n.

The most convenient method of solution is to find, first l, then n.

Substituting in the formula, $s=\dfrac{rl-a}{r-1}$,

$$\frac{91}{162}=\frac{-\frac{1}{3}l-\frac{3}{4}}{-\frac{1}{3}-1}, \quad \text{whence,} \quad l=-\frac{1}{324}. \quad \textit{Ans.}$$

Using $l=ar^{n-1}$, $-\frac{1}{324}=\frac{3}{4}(-\frac{1}{3})^{n-1}$.

Whence, $-\frac{1}{324}\times\frac{4}{3}=(-\frac{1}{3})^{n-1}$, and $n=6$. *Ans.*

Let the pupil check these results.

EXERCISE 105

Find the first term and the sum when

1. $n=6$, $r=3$, $l=486$. **2.** $n=8$, $r=-2$, $l=-640$.

Find the ratio when

3. $a = -2$, $l = 2048$, $n = 6$. **4.** $a = 9$, $l = \frac{16}{9}$, $s = 23\frac{4}{9}$.

Find the number of terms when

5. $a = 2$, $r = 2$, $s = 62$. **6.** $a = \frac{4}{3}$, $l = \frac{1}{12}$, $r = \frac{1}{2}$.

How many consecutive terms must be taken from the series

7. 2, 4, 8, . . . to make the sum 62?

8. $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, . . . to make the sum $\frac{211}{108}$?

125. Geometrical Means.

Ex. Insert 5 geometrical means between 3 and $\frac{1}{243}$.

We have given $a = 3$, $l = \frac{1}{243}$, $n = 7$, to find r.

Solving by § 124, $r = \frac{1}{3}$.

Hence, the required geometrical means are

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}. \quad \textit{Ans.}$$

In case only one geometrical mean is to be inserted between two quantities, a and b, this one mean is found most readily by using the formula $\sqrt{ab}$. For if x represents the geometrical mean between a and b, the series will be

$$a, \quad x, \quad b.$$

Hence, $\frac{x}{a} = \frac{b}{x}$. $\therefore x^2 = ab$, $x = \sqrt{ab}$.

EXERCISE 106

Insert

1. Three geometrical means between 8 and $\frac{1}{2}$.

2. Three geometrical means between $\frac{9}{4}$ and $\frac{4}{9}$.

3. Six geometrical means between $\frac{1}{24}$ and $-\frac{16}{3}$.

4. Four geometrical means between $-\frac{7}{2}$ and 3584.

5. Six geometrical means between 56 and $-\frac{7}{16}$.

Find the geometrical mean between

6. $4\frac{1}{6}$ and $\frac{3}{8}$.

7. $28\,a^3x$ and $63\,axy^4$.

8. $\dfrac{a\sqrt{x}}{c^2\sqrt[3]{y}}$ and $\dfrac{y\sqrt[3]{a^2}}{x\sqrt{c^3}}$.

9. .7 and 3.43.

10. .005 and .125.

11. $5\sqrt{2}+1$ and $5\sqrt{2}-1$.

12. Insert 7 geometrical means between $\dfrac{8}{n^2}$ and $\dfrac{n^2}{2}$.

13. Is a mean proportional between two numbers the same as the geometric mean between the numbers?

126. Miscellaneous Problems.

Ex. Find four numbers in G. P. such that the sum of the first and fourth is 56, and of the second and third is 24.

Denote the required numbers by a, ar, ar^2, ar^3.

Then $$a+ar^3=56.$$
$$ar+ar^2=24.$$

Or $$a(1+r^3)=56. \quad \dots\dots\dots \quad (1)$$
$$ar(1+r)=24. \quad \dots\dots\dots \quad (2)$$

Divide (1) by (2), $$\frac{1-r+r^2}{r}=\frac{7}{3}.$$

Hence, $$3-3\,r+3\,r^2=7\,r.$$
$$3\,r^2-10\,r=-3.$$
$$r=3,\ \tfrac{1}{3}.$$

And $$a=2,\ 54.$$

Hence, the numbers are 2, 6, 18, 54.
Or 54, 18, 6, 2. *Ans.*

EXERCISE 107

Find the first two terms of the series in which

1. The 3d term is 2, and the 5th is 18.

2. The 4th term is $\frac{3}{2}$ and the 9th is 48.

3. The 5th term is 6 and the 11th is $\frac{3}{32}$.

Determine the nature, whether arithmetic or geometric, of each of the following series:

4. $\frac{1}{4}, \frac{1}{3}, \frac{4}{9}, \ldots$

5. $\frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \ldots$

6. $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \ldots$

7. $7\frac{1}{3}, 5\frac{7}{8}, 4\frac{5}{12}, \ldots$

8. Divide 65 into 3 parts in geometrical progression, such that the sum of the first and third is $3\frac{1}{3}$ times the second part.

9. There are 3 numbers in G. P. whose sum is 49. The sum of the first and second of these numbers is to the sum of the first and third as 3 to 5. Find the numbers.

10. Find four numbers in G. P., such that the sum of the first and third is 10, and of the second and fourth 30.

11. Three numbers whose sum is 24 are in A. P., but if 3, 4 and 7 are added to them respectively, the sums obtained will be in G. P. Find the numbers.

12. The sum of $225 was divided among four persons in such a manner that the shares were in G. P., and the difference between the greatest and least was to the difference between the means as 7 is to 2. Find each share.

13. Find the sum of $\frac{1}{\sqrt{2}-1}, \sqrt{2}, \frac{2}{\sqrt{2}+1}, \ldots$ *ad infinitum.*

14. There are four numbers, the first three of which are in G. P., and the last three are in A. P.; the sum of the first and last is 14, and of the means is 12. Find the numbers.

15. If the series $\frac{3}{4}, \frac{1}{2}, \ldots$ is arithmetical, find the 102d term; if geometrical, find the sum to infinity.

16. Insert between 2 and 9 two numbers, such that the first three of the four may be in A. P., and the last three in G. P.

17. Prove that the series $\sqrt{2}-1$, $3\sqrt{2}-4$, $2(5\sqrt{2}-7)$, $\ldots$ is geometrical; that its ratio is $2-\sqrt{2}$; and that its sum to infinity is unity.

18. On p. 23 a table gives the amount of $1 in different periods of time at simple interest and also at compound interest. Which of these three series of numbers forms an A. P. and which a G. P.?

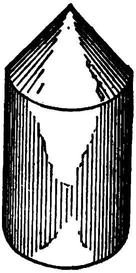
19. If an air pump at each stroke removes $\frac{1}{8}$ of the air in a receiver, what fraction of the air is left at the end of 10 strokes?

20. If the amount of air in a receiver is indicated by the height of a mercury column in a tube attached to the receiver, and this height is 30 in. at the start, what will the height of the mercury be at the end of the 10 strokes mentioned in Ex. 19?

21. There were 2,500,000,000,000 tons of coal in the United States in the year 1910, and 3,000,000,000 tons were consumed between the year 1900 and 1910. If the consumption of coal should double every decade, tell to the nearest decade how long the coal in the United States would last.

EXERCISE 108

1. A cylindrical cistern is d feet in diameter and h feet deep. Obtain a formula for G the number of gallons the cistern will hold if $7\frac{1}{2}$ gallons are taken as equivalent to 1 cu. ft.



2. Find the same if 1 gal. is taken as 231 cu. in.

3. An oil can has the shape of a cylinder topped by a cone. If the diameter is d inches, the height of the cylinder h inches, and the height of the cone c inches, and 1 gal. $=231$ cu. in., obtain a formula for the number of gallons of oil the can will hold.

4. A rectangular tank is l feet long and w feet wide. Water is running into the tank at the rate of c cubic feet per minute. Obtain a formula for g, the number of inches. (I) the water will rise in the tank per hour.

5. Find the same if, instead of c cubic feet, the water is flowing into the tank through a circular opening d inches in diameter at the rate of f feet per minute.

6. In selling a certain article, a merchant wishes to fix an asking price such that he can give two successive discounts of 20 % and 10 % and still make a profit of 40 % on the article. If c is the cost price, a the asking price, and s the selling price, find a formula for a in terms of c.

7. Work Ex. 6, after changing 20 % into d_1, 10 % into d_2, and 40 % into g. In solving this problem what letter was eliminated between what two formulas?

8. If the dimensions of a school room are l, w, h feet, and 200 cu. ft. of space are allowed for each pupil, the number of pupils (P) allowable in the room is given by the formula $P=\frac{lwh}{200}$. Express this formula as a rule.

9. By use of the formula given in Ex. 8, determine the amount of floor space a room should have, to accommodate 36 pupils, if the room is 12 ft. high.

CHAPTER XI

LOGARITHMS

127. The Logarithm of a number is the exponent of that power of another number, taken as the base, which equals the given number.

Thus, $1000=10^3$. Hence, $\log 1000=3$, 10 being taken as the base.

Again, if 8 is taken as the base, $4=8^{\frac{2}{3}}$. Hence, $\log 4=\frac{2}{3}$.

If 5 is taken as the base, $\log 125=3$, $\log \frac{1}{25}=\log 5^{-2}=-2$, etc.

The base is sometimes stated as above; but when desirable, it is indicated by writing it as a small subscript to the word *log*.

Thus, the above expressions might be written,

$$\log_{10} 1000=3;\quad \log_8 4=\tfrac{2}{3};\quad \log_5 125=3;\quad \log_5 \tfrac{1}{25}=-2;\quad \text{etc.}$$

In general, by the definition of a logarithm,

$$\text{number}=(\text{base})^{\text{logarithm}},$$

or $$N=B^l.\quad \text{Hence, } \log_B N=l.$$

Hence, logarithms are simply special forms of exponents.

Ex. 1. Express $2^3=8$ in the logarithmic form.

We have, $\log_2 8=3$. *Ans.*

Ex. 2. Express $\log_2 16=4$ in the exponential form.

We have $2^4=16$. *Ans.*

Ex. 3. If the base is 3, what is the log of $\frac{1}{81}$?

$$\frac{1}{81}=\frac{1}{3^4}=3^{-4}.$$

$$\therefore \log_3 \frac{1}{81} = -4. \quad \textit{Ans.}$$

Ex. 4. If the base is 10, what is the log of .001?

$$.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}.$$

$$\therefore \log .001 = -3. \quad \textit{Ans.}$$

When 10 is the base, it is the custom not to write it as a subscript, and to leave the place for the subscript blank. See the preceding example.

EXERCISE 109

1. Express the following in logarithmic form: $3^2=9$; $2^4=16$; $4^3=64$; $9^2=81$; $2^2=4$; $3^3=27$; $a^b=c$; $4^0=1$; $a^0=1$; $8^{\frac{2}{3}}=4$; $9^{\frac{3}{2}}=27$.

2. Express the following in exponential form: $\log_3 81=4$; $\log_2 32=5$; $\log_8 64=2$; $\log_9 81=2$; $\log_6 216=3$; $\log_a Q=c$; $\log_4 1=0$; $\log_b 1=0$; $\log_{25} 125=\frac{3}{2}$; $\log_{16} 8=\frac{3}{4}$.

3. If the base is 2, state the logarithm of 2; 4; 32; 128; $\frac{1}{4}$; $\frac{1}{16}$; $\frac{1}{64}$; 1.

4. If the base is 4, give the log of 1024; 64; 16; 1; 16; $\frac{1}{4}$.

5. If the base is 10, state the log of 10; 1000; 100; 10,000; 1; .1; .01; .0001; .001.

6. Simplify $\log_3 9+\log_3 27+\log_4 64+\log_4 \frac{1}{16}+\log_3 \frac{1}{9}+\log_{10} \frac{1}{10}$.

7. Simplify $\log_2 32-\log_2 \frac{1}{32}+\log_4 8-\log_8 16$.

8. Simplify $\log 3 \log_2 4+2 \log_3 9+\log_{10} .1-\log_3 \frac{1}{9}$.

9. If the base is 10, what is the log of 10? Of 100? Hence, the log of any number between 10 and 100 lies between what two consecutive whole numbers?

10. If the base is 10, what is the log of 1000? Hence, the log of any number between 100 and 1000 lies between what two consecutive whole numbers?

11. The log of any number between 1000 and 10,000 lies between what two consecutive whole numbers?

12. Show that $\log \sqrt{10} = .5$. That $\log \sqrt{1000} = 1.5$. That $\log \sqrt{.1} = -.5$. That $\log \sqrt[3]{100} = .6667$.

128. Uses of Logarithms. One of the principal uses of logarithms is to simplify numerical work. For instance, by logarithms the numerical work of *multiplying* two numbers is converted into the simpler work of *adding* the logarithms of these numbers.

To illustrate this principle, we may take the simple case of multiplying two numbers which are exact powers of 10, as 1000 and 100. Thus,

$$1000 = 10^3$$

$$100 = 10^2$$

Hence, $$1000 \times 100 = 10^5 = 100{,}000,$$

the multiplication being performed by the addition of exponents.

Similarly, if $384 = 10^{2.58433+}$

and $25 = 10^{1.39794+}$.

To multiply 384 by 25,

Add the exponents of $10^{2.58433+}$ and $10^{1.39794+}$ thus obtaining $10^{3.98227+}$.

Then get from a table of logarithms the value of $10^{3.98227+}$: viz., 9600.

In like manner, by the use of logarithms, the process of dividing one number by another is converted into the simpler process of subtracting one exponent, or log, from another. The process of involution, also, is converted into the simpler process of multiplication; and the extraction of a root into the simpler process of division.

We can save labor still further, through the use of logarithms, by committing to memory the logs of numbers that are frequently used, as of

$$2, 3, \ldots 9, \pi, \sqrt{\pi}, \frac{1}{\pi}, \sqrt{2}, \sqrt{3}, \text{etc.}$$

By the use of the *slide rule*, the practical use of logarithms is reduced to sliding one rod along another and reading off the number at one end of a rod.

It will be a useful exercise to teach the class the use of the slide rule in connection with the study of this chapter.

EXERCISE 110

1. Write out the value of each power of 2 up to 2^{20} in the form of a table.

Thus, $2^1=2$, $2^2=4$, $2^3=8$, etc.

2. By means of this table, multiply 32 by 8, performing the multiplication by the addition of exponents.

3. In like manner, convert each of the following multiplications into an addition: 32×16, 64×32, 1024×16, 512×64.

4. Convert each of the following divisions into a subtraction: $1024\div16$, $512\div64$, $32{,}768\div1024$.

5. Convert each of the following involutions into a multiplication: $(32)^3$, $(64)^2$, $(32)^4$.

6. Convert each of the following root extractions into a division: $\sqrt[3]{64}$, $\sqrt[5]{1024}$, $\sqrt[4]{4096}$.

7. If the base is 10, what is the log of 1? Of .1? Hence, the log of any number between 1 and .1 (as of .7) lies between what two consecutive whole numbers?

8. If the base is 10, what is the log of .01? Hence, the log of any number between .1 and .01 (as of .07) lies between what two consecutive whole numbers?

9. The log of any number between .01 and .001 (as of .007) lies between what two consecutive whole numbers?

129. Characteristic and Mantissa. If a given number, as 384, is not an exact power of the base, its logarithm, as 2.58433^{+}, consists of two parts: the whole number, called the *characteristic*, and the decimal part, called the *mantissa*.

To obtain a rule for determining the characteristic of a given number (the base being 10), we have:

$$10{,}000 = 10^4, \quad \text{hence} \quad \log 10{,}000 = 4;$$
$$1000 = 10^3, \quad \text{hence} \quad \log 1000 = 3;$$
$$100 = 10^2, \quad \text{hence} \quad \log 100 = 2;$$
$$10 = 10^1, \quad \text{hence} \quad \log 10 = 1.$$

Hence, any number between 1000 and 10,000 has a logarithm between 3 and 4; that is, its log is 3 plus a fraction.

But every integral number between 1000 and 10,000 contains four digits. Hence, every integral number containing *four* figures has 3 for a characteristic.

Similarly, every number between 100 and 1000 (i.e., a number containing *three* figures to the left of the decimal point) has 2 for a characteristic.

A number between 10 and 100 (i.e., a number containing *two* integral figures) has 1 for a characteristic.

Every number between 1 and 10 (i.e., every number containing *one* integral figure) has 0 for a characteristic.

Hence, *the characteristic of an integral or mixed number is one less than the number of figures to the left of the decimal point.*

130. Characteristic of a Decimal Fraction.

$$1 = 10^0. \qquad \therefore \log 1 = 0;$$

$$.1 = \frac{1}{10} = 10^{-1}. \qquad \therefore \log .1 = -1;$$

$$.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}. \qquad \therefore \log .01 = -2;$$

$$.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}. \qquad \therefore \log .001 = -3, \text{ etc.}$$

Hence, the logarithm of any number between .1 and 1 (as of .4, for instance) will lie between −1 and 0, and hence will consist of −1 plus a positive fraction.

The logarithm of every number between .01 and .1 (as of

.0372, for instance) will be between -2 and -1, and hence will consist of -2 plus a positive fraction; and so on.

Hence, *the characteristic of a decimal fraction is negative, and is numerically one more than the number of zeros between the decimal point and the first significant figure.*

There are two ways in common use for writing the characteristic of a decimal fraction.

Thus, (1) log .0384 = $\bar{2}$.58433, the minus sign being placed over the characteristic 2, to show that it alone is negative, the mantissa being positive.

Or (2) 10 is added to and subtracted from the log, giving

$$\log .0384 = 8.58433 - 10.$$

In practice, the following rule is used for determining the characteristic of the logarithm of a decimal fraction:

Take the number of zeros between the decimal point and the first significant figure; subtract it from 9, and annex −10 after the mantissa.

EXERCISE 111

Give the characteristic of

1. 452.
2. 16,730.
3. 767.5.
4. 64.56.
5. 9.22678.
6. .08267.
7. 1.0042.
8. 7.92631.
9. .007.
10. .0000625.
11. 7.
12. 6267.3.
13. .000227.
14. 100.58.
15. 23.7621.

16. How many figures to the left of the decimal point (or how many zeros immediately to the right) are there in a number, the characteristic of whose logarithm is 3? 2? 5? 1? 0? 4? 8−10? 7−10? 9−10?

17. Can you make up a rule for fixing the decimal point in that number which corresponds to a given logarithm?

18. If log 632 = 2.8007, express 632 as a power of 10.

19. If $257 = 10^{2.4099}$, what is the log of 257?

20. If a number lies between 10,000 and 20,000, what will its characteristic be?

21. If a number lies between 10,000 and 100,000, between what two integral numbers must its logarithm lie?

131. Mantissas of numbers are computed by methods, usually algebraic, which lie outside the scope of this book. After being computed, the mantissas are arranged in tables, from which they are taken when needed. In this connection, it is important to note that

The position of the decimal point in a number affects only the characteristic, not the mantissa, of the logarithm of the number.

Thus, if $\log 6754 = 3.82956$

$$\log 67.54 = \log \frac{6754}{100} = \log \frac{10^{3.82956}}{10^2} = \log 10^{1.82956} = 1.82956.$$

In general,

$$\begin{aligned} \log 6754 &= 3.82956. \\ \log 675.4 &= 2.82956. \\ \log 67.54 &= 1.82956 \\ \log 6.754 &= 0.82956 \\ \log .6754 &= 9.82956 - 10. \\ \log .06754 &= 8.82956 - 10, \text{ etc.} \end{aligned}$$

132. Direct Use of a Table of Logarithms; that is, *given a number, to find its logarithm from a table.*

On pages 218–219 is given a table of mantissas for numbers from 1 to 1000. These mantissas are carried out to four decimal places; hence this table is called a four-place table. (As a matter of convenience the decimal points are omitted in printing this table.)

In the table, the left-hand column is a column of numbers, and is headed N.

The mantissa of each of these numbers is the number opposite it in the next column to the right. In the top row of each page are the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

To obtain the mantissa for a number of three figures, as 364, we take 36 in the first column, and look along the row beginning with 36 till we come to the column headed 4. The mantissa thus obtained is .5611.

If the number whose mantissa is sought contains four or five figures,

Obtain from the table the mantissa for the first three figures, and also that for the next higher number, and subtract;

Multiply the difference between the two mantissas by the fourth (*or fourth and fifth*) *figure expressed as a decimal;*

And ADD *the result to the mantissa for the first three figures.*

Thus, to find the mantissa for 167.49,

$$\begin{aligned} \text{Mantissa for } 168 &= .2253 \\ \text{Mantissa for } 167 &= \underline{.2227} \\ \text{Difference} &= .0026 \end{aligned}$$

Since an increase of 1 in the number (from 167 to 168) makes an increase of .0026 in the mantissa, an increase of .49 of 1 in the number will make an increase of .49 of .0026 in the mantissa.

But $.0026 \times .49 = .001274$ or $.0013-$.

Hence,

$$\begin{aligned} &.2227 \\ &\underline{13} \\ \text{Mantissa for } 167.49 = &.2240 \end{aligned}$$

Hence, to obtain the logarithm of a given number,

Determine the characteristic by § 129 *or* § 130;

Neglect the decimal point in the given number, and obtain from the table (pp. 218, 219) *the mantissa for the given figures.*

Exs. Log 52.6 = 1.7210. Log .00094 = 6.9731 − 10.
Log 167.49 = 2.2240. Log .042308 = 8.6264 − 10.

N.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	414	453	492	531	569	607	645	682	719	755
12	792	828	864	899	934	969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	335	367	399	430
14	461	492	523	553	584	614	644	673	703	732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	279
17	304	330	355	380	405	430	455	480	504	529
18	553	577	601	625	648	672	695	718	742	765
19	788	810	833	856	878	900	923	945	967	989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	222	243	263	284	304	324	345	365	385	404
22	424	444	464	483	502	522	541	560	579	598
23	617	636	655	674	692	711	729	747	766	784
24	802	820	838	856	874	892	909	927	945	962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	183	200	216	232	249	265	281	298
27	314	330	346	362	378	393	409	425	440	456
28	472	487	502	518	533	548	564	579	594	609
29	624	639	654	669	683	698	713	728	742	757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	914	928	942	955	969	983	997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	145	159	172
33	185	198	211	224	237	250	263	276	289	302
34	315	328	340	353	366	378	391	403	416	428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	563	575	587	599	611	623	635	647	658	670
37	682	694	705	717	729	740	752	763	775	786
38	798	809	821	832	843	855	866	877	888	899
39	911	922	933	944	955	966	977	988	999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	128	138	149	160	170	180	191	201	212	222
42	232	243	253	263	274	284	294	304	314	325
43	335	345	355	365	375	385	395	405	415	425
44	435	444	454	464	474	484	493	503	513	522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	628	637	646	656	665	675	684	693	702	712
47	721	730	739	749	758	767	776	785	794	803
48	812	821	830	839	848	857	866	875	884	893
49	902	911	920	928	937	946	955	964	972	981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	093	101	110	118	126	135	143	152
52	160	168	177	185	193	202	210	218	226	235
53	243	251	259	267	275	284	292	300	308	316
54	324	332	340	348	356	364	372	380	388	396
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	482	490	497	505	513	520	528	536	543	551
57	559	566	574	582	589	597	604	612	619	627
58	634	642	649	657	664	672	679	686	694	701
59	709	716	723	731	738	745	752	760	767	774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	853	860	868	875	882	889	896	903	910	917
62	924	931	938	945	952	959	966	973	980	987
63	993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	069	075	082	089	096	102	109	116	122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	195	202	209	215	222	228	235	241	248	254
67	261	267	274	280	287	293	299	306	312	319
68	325	331	338	344	351	357	363	370	376	382
69	388	395	401	407	414	420	426	432	439	445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	513	519	525	531	537	543	549	555	561	567
72	573	579	585	591	597	603	609	615	621	627
73	633	639	645	651	657	663	669	675	681	686
74	692	698	704	710	716	722	727	733	739	745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	808	814	820	825	831	837	842	848	854	859
77	865	871	876	882	887	893	899	904	910	915
78	921	927	932	938	943	949	954	960	965	971
79	976	982	987	993	998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	085	090	096	101	106	112	117	122	128	133
82	138	143	149	154	159	165	170	175	180	186
83	191	196	201	206	212	217	222	227	232	238
84	243	248	253	258	263	269	274	279	284	289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	345	350	355	360	365	370	375	380	385	390
87	395	400	405	410	415	420	425	430	435	440
88	445	450	455	460	465	469	474	479	484	489
89	494	499	504	509	513	518	523	528	533	538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	590	595	600	605	609	614	619	624	628	633
92	638	643	647	652	657	661	666	671	675	680
93	685	689	694	699	703	708	713	717	722	727
94	731	736	741	745	750	754	759	763	768	773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	823	827	832	836	841	845	850	854	859	863
97	868	872	877	881	886	890	894	899	903	908
98	912	917	921	926	930	934	939	943	948	952
99	956	961	965	969	974	978	983	987	991	996
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

EXERCISE 112

Find the logarithms of the following numbers:

1. 37.
2. 85.
3. 6.
4. 90.
5. 300.
6. 175.
7. 32.9.
8. 4.75.
9. .08.
10. 1.02.
11. .0758.
12. 5780.
13. .00217.
14. 63.21.
15. 3.002.
16. .7788.
17. .04275.
18. 234.76.
19. 5.6107.
20. 7781.4.

133. Inverse Use of a Table of Logarithms; that is, *given a logarithm, to find the number,* termed **antilogarithm,** *corresponding to this logarithm.*

The method is as follows:

From the table, find the figures corresponding to the mantissa of the given logarithm.

Use the characteristic of the given logarithm to fix the decimal point of the figures obtained.

Ex. Find the antilogarithm of 1.5658, that is, find the number whose logarithm is 1.5658.

The figures corresponding to the mantissa, .5658, are 368.

Since the characteristic is 1, there are 2 figures at the left of the decimal point.

Hence, Antilog $1.5658 = 36.8$.

In case the given mantissa does not occur in the table, *obtain from the table the next lower mantissa with the corresponding three figures of the antilogarithm;*

Subtract the tabular mantissa from the given mantissa;

Divide this difference by the difference between the tabular mantissa and the next higher mantissa in the table;

Annex the quotient to the three figures of the antilogarithm obtained from the table.

Ex. Find antilog 2.4237.

.4237 does not occur in the table, and the next lower mantissa is .4232. The difference between .4232 and .4249 is .0017.

Hence, we have antilog 2.4237 = 265.29

$$\frac{4232}{17)5.00(.29}$$

For if a difference of 17 in the last two figures of the mantissa makes a difference of 1 in the third figure of the antilog, a difference of 5 in the mantissa will make a difference of $\frac{5}{17}$ of 1 or .29 with respect to the third figure of the antilog.

EXERCISE 113

Find the numbers corresponding to the following logarithms:

1. 1.6335.
2. 2.8865.
3. 2.3729.
4. 0.5775.
5. 3.9243.
6. 1.8476.
7. 0.6117.
8. 9.7973 − 10.
9. 7.9047 − 10.
10. 8.6314 − 10.
11. 7.7007 − 10.
12. 6.1004 − 10.
13. 0.4133.
14. 1.4900.
15. 3.8500.
16. 1.8904.
17. 2.4527.
18. 9.6402 − 10.

19. Write $\log 17 = 1.2304$ as a number equal to a power of 10.

134. Properties of Logarithms. As a result of the properties of exponents stated in § 57 (p. 111), it follows that

(1) $\log ab = \log a + \log b$ (3) $\log a^p = p \log a$

(2) $\log\left(\frac{a}{b}\right) = \log a - \log b$ (4) $\log \sqrt[p]{a} = \frac{\log a}{p}$.

PROOF:

$$\begin{array}{lll} \text{Let } a = 10^m. & \therefore \log a & = m. \\ b = 10^n. & \therefore \log b & = n. \\ ab = 10^{m+n}. & \therefore \log ab & = m + n = \log a + \log b. \quad . \; . \; . \quad (1) \\ \frac{a}{b} = 10^{m-n}. & \therefore \log\left(\frac{a}{b}\right) & = m - n = \log a - \log b. \; . \quad . \; . \quad (2) \end{array}$$

$$a^p = 10^{pm}. \qquad \therefore \log a^p = pm = p \log a. \quad \ldots \quad (3)$$

$$\sqrt[p]{a} = 10^{\frac{m}{p}}. \qquad \therefore \log \sqrt[p]{a} = \frac{m}{p} = \frac{\log a}{p}. \quad \ldots \quad (4)$$

The same properties may be proved in like manner for a system of logarithms with any other base than 10.

135. Properties Utilized for Purposes of Computation.

I. To Multiply Numbers,

Add their logarithms, and find the antilogarithm of the sum. This will be the product of the numbers.

II. To Divide One Number by Another,

Subtract the logarithm of the divisor from the logarithm of the dividend, and obtain the antilogarithm of the difference.

III. To Raise a Number to a Required Power,

Multiply the logarithm of the number by the index of the power. Find the antilogarithm of the product.

IV. To Extract a Required Root of a Number,

Divide the logarithm of the number by the index of the required root. Find the antilogarithm of the quotient.

Ex. 1. Multiply 527 by .083 by the use of logs.

$$\begin{aligned} \log\ 527 &= 2.7218 \\ \log\ .083 &= \underline{8.9191} - 10 \\ \text{antilog}\ 1.6409 &= 43.7^{+}. \quad \textit{Product.} \end{aligned}$$

Ex. 2. Compute the amount of \$1 at 6 % for 20 years at compound interest.

The amount of \$1 at 6% for 20 years $= (1.06)^{20}$.

$$\begin{aligned} \log 1.06 &= .0253 \\ &\underline{\quad\ \ 20} \\ \text{antilog}\ 0.5060 &= 3.21^{-} \end{aligned}$$

$\therefore$ \$3.21. *Ans.*

Computing $(1.06)^{20}$ by direct multiplication, will make clear the amount of labor sometimes saved by the use of logarithms.

Ex. 3. Extract the 7th root of 15.

$$\begin{aligned} \log 15 &= 1.1761 \\ \tfrac{1}{7}\log 15 &= 0.1680 \\ \text{antilog } 0.1680 &= 1.47^{+}. \quad \textit{Ans.} \end{aligned}$$

Ex. 4. By use of logarithms compute $\dfrac{8.4\times 32.4}{2\sqrt{576}\times 3.78}$.

$$\begin{array}{rrcrr} \log 8.4 = & 0.9243 & \qquad & \log 2 = & 0.3010 \\ \log 32.4 = & \underline{1.5105} & & \log \sqrt{576} = & 1.3802 \\ & 2.4348 & & \log 3.78 = & \underline{0.5775} \\ & \underline{2.2587} & & & 2.2587 \\ \text{Antilog } & 0.1761 = 1.5. \quad \textit{Ans.} & & & \end{array}$$

EXERCISE 114

Find, by use of logarithms, the approximate value of

1. 75×1.4.
2. 9.8×3.5.
3. $15.1\times .235$.
4. $831\times .25$.
5. $\dfrac{54.7}{13.4}$.
6. $\dfrac{336.8}{7984}$.
7. $\dfrac{.317}{.0049}$.
8. $\dfrac{-78.9}{98.7}$.
9. $\dfrac{42.316}{.06214}$.
10. $.48\div(-1.79)$.
11. $\dfrac{1.78\times 19}{23.7}$.
12. $\dfrac{3.51\times 67}{97.7}$.
13. $\dfrac{12.9}{4.7\times 9.1}$.
14. $47.1\times 3.56\times .0079$.
15. $.0631\times 7.208\times .51272$.
16. $4.77\times(-.71)\div(.83)$.
17. $\dfrac{523\times 249}{767\times 396}$.

18. $(2.3)^3$.

19. $(1.032)^{15}$.

20. $(3.57)^4$.

21. $(.96)^7$.

22. $(.795)^6$.

23. $\sqrt{19}$.

24. $\sqrt[3]{3.29}$.

25. $\sqrt[5]{7.65}$.

26. $\sqrt{.44}$.

27. $\sqrt[3]{61}$.

28. $\sqrt{.00429}$.

29. $(2.91)^{\frac{5}{3}}$.

30. $\sqrt[4]{10^3}$.

31. $\sqrt{30}\times\sqrt[3]{-5}$.

32. $\sqrt[3]{19}\div\sqrt{46}$.

33. $\sqrt[4]{\dfrac{52{,}900}{67\times 5.18}}$.

34. $\sqrt{\dfrac{37.56\times 26.5}{22.7\times 16.78}}$.

By the use of logarithms:

35. Find the amount of $1250 at 6 per cent compound interest for 12 years. Also make the computation without the use of logarithms. What fraction of the work is saved by the use of logarithms?

36. Find the amount of $25 at 5 per cent compound interest for 500 years.

37. Find the amount of $300 at 6 per cent for 50 years, interest being compounded semi-annually.

38. Find the amount of $300 at 6 per cent for 50 years, interest being compounded quarterly.

39. Find to the nearest thousandth of a foot the radius of a circle whose area is 100 sq. yd. (Use $\pi=\frac{22}{7}$.)

40. Find the radius of a sphere whose volume is 20 cu. ft. (Use $V=\frac{4}{3}\pi R^3$ and $\pi=\frac{22}{7}$.)

41. A given parallelogram is 12.7 ft. long and 8.9 ft. high. Find the side of a square whose area is equal to that of the parallelogram.

42. Compute $\sqrt[3]{15}$ to three decimal places without the use of logarithms. Now obtain the same result by the use of logarithms. Compare the amount of work in the two processes.

43. Find log $\sqrt[3]{10}\times\sqrt[5]{100}$ without the use of tables.

44. By use of logarithms, find the value of $\sqrt{b^2-c^2}$, when $b=276.5$, $c=172.4$.

45. How many years will it take a sum of money to double itself at 5 % compound interest? At 7 per cent?

46. A certain sum of money at compound interest doubles itself in 10 years. Find the rate of interest.

47. Find the product of 1.7^4, .38, and 24.6 by use of logs.

48. Given that $\log x=3.7154$, find $\log \sqrt{x}$.

49. Given $\log 2=0.3010$ and $\log 3=0.4771$, find log 12 without using the tables.

By use of logs, compute the value of

50. $a\sqrt{\frac{Cr}{n}}$, when $a=15$, $C=127$, $r=1.86$, $n=32.17$.

51. $\sqrt{32.2\,d\left(1+\frac{3\,h}{a}\right)}$, when $d=27$, $h=16$, $a=5.23$.

52. $\frac{(P-t)(T-t)}{P-T}\times .0045\,C$, when $P=120$, $t=40.5$, $T=61.5$, $C=3560$.

136. Exponential Equations. An exponential equation is an equation of the form $a^x=b$, where a and b are known quantities and x is unknown.

An equation of this kind can be solved by taking the logarithm of each member.

Ex. 1. Solve $(1.06)^x=3$ (that is, find the number of years in which $1 will amount to $3 at 6 % compound interest).

$$x \log 1.06=\log 3.$$

$$x=\frac{\log 3}{\log 1.06}=\frac{.4771}{.0253}=18.858^{-}\ \textit{Years.}$$

Ex. 2. Given $.3^x=2$; find the value of x.

Taking the logarithm of each member of the equation,

$$x \log (.3) = \log 2.$$

Hence,

$$x = \frac{\log 2}{\log .3} = \frac{.3010}{9.4771-10}.$$

$$= \frac{.3010}{-.5229} = -.576-. \quad \textit{Ans.}$$

EXERCISE 115

Find the approximate value of x in each of the following exponential equations:

1. $40^x = 75.$
2. $7^x = 25.$
3. $5^{x+1} = 17.$
4. $1.3^x = 7.2.$
5. $3^{x-2} = 5.$
6. $(.4)^{-x} = 7.$
7. $20^x = 5.$
8. $27^{2x-1} = 8.$
9. $.703^x = 1.09.$

10. In how many years will $4000 amount to $8500 at 6 per cent, if interest is compounded annually? Semi-annually?

11. In how many years will $675.50 amount to $2000 at 5 per cent compound interest?

12. In how many years will $200 amount to $1800 at 6 per cent, if interest is compounded quarterly?

13. If $1280 amounts to $37,770 in 50 years at compound interest, what is the rate per cent?

EXERCISE 116

1. The costs in dollars per month of running two auto trucks of the same size for the first six months of a year, and the number of miles run per month were as follows:

		JAN.	FEB.	MAR.	APRIL	MAY	JUNE
First Truck	Cost	$32	$33	$43	$31	$37	$29
	Miles run	627	473	456	608	712	802
Second Truck	Cost	$28	$36	$49	$33	$31	$39
	Miles run	732	884	672	516	416	523

On the same diagram construct an efficiency curve (but no other curve) for each truck. From the chart determine which of the two trucks was more efficient in the winter months. Also which was more efficient in the spring months.

2. Graph $y=2x+b$ when $b=1$. On the same diagram, graph $y=2x+b$ when $b=2$. When $b=-1$. When $b=0$.

3. On the same diagram, graph $y=x^2+b$ when $b=0$, 1, 3, -2.

Graph the following in each case selecting the most appropriate form of graph:

4. In a given year the total profits of a department store were $32,316. The respective profits of the five departments in the store were as follows: $6820, $3127. $7180, $10,173, $5016.

5. In the year 1830 a bushel of wheat, on the average in the United States, was raised by 3½ hours of labor by one man; in the year 1910, a bushel was raised by 10 minutes of labor.

6. The following tabulation gives the population of the United States in millions, and the production of pig iron in millions of tons in the years specified:

YEAR	1870	1880	1890	1900	1910	1920
Population	39	50	63	76	92	
Pig iron	1.8	4.3	9	13.8	27.3	
Pounds per capita						

7. The following tabulation gives the quota assigned to the salesman of a certain article for eight months and also the number of articles actually sold by him:

MONTH	JAN.	FEB.	MAR.	APRIL	MAY	JUNE	JULY	AUGUST
Quota	200	210	220	230	240	240	240	240
Sales	176	183	201	240	262	272	289	293

What can you learn from your chart?

EXERCISE 117

GENERAL REVIEW

1. Factor:

(1) $(3x^2-2x)^2+5(3x^2-2x)-6.$ (4) $8a^3+y^6.$
(2) $ax+y-by+ay+x-bx.$ (5) x^2-16x^6
(3) $x^4+6x^3+9x^2-4x-12.$ (6) $1-27a^3x^6$

2. Simplify $\frac{2x+1}{3x-5}-\frac{5x-6}{2x+7}-\frac{11x^2-22x+10}{35-6x^2}$ and check by letting $x=1$.

3. Solve $h=\frac{s^2}{4(2r-h)}$ for h.

4. Solve $\sqrt{3x+10}-\sqrt{10x+16}-\sqrt{x+2}=0$ and check.

5. Solve $x-y=1$, $\frac{x}{y}-\frac{y}{x}=\frac{5}{6}$.

6. Find two numbers such that their sum multiplied by the greater is 126, and their difference multiplied by the less is 20.

7. Without solving or substituting, determine whether 2 and $-.75$ are the roots of $x^2-15=2x$. Also whether $\frac{1}{2}(3\pm\sqrt{3})$ are the roots of $2x^2+3-6x=0$.

8. Insert four geometric means between 160 and 5.

9. How many terms of the A. P. 42, 39, 36, . . . must be taken to make 315?

10. Simplify $\frac{1}{a-\frac{1}{a}}+\frac{1}{a+\frac{1}{a}}-\frac{2}{a-\frac{1}{a^3}}+\frac{b}{ab+\frac{b}{a}}$ and check by letting $a=2$ and $b=1$.

11. Solve for x and y: $mx+\frac{n}{y}=1$, $nx+\frac{m}{y}=1$.

12. If A gives B \$6, B will have $\frac{2}{3}$ as much as A has left. But if B gives A \$5, B will have $\frac{3}{5}$ as much as A then has. How much does each man have?

13. Denoting the first of three consecutive numbers by n, prove that the sum of any three consecutive numbers is equal to three times the middle of the three numbers.

14. Two men, A and B, together do a piece of work in 12 days. B alone would need 10 days more than A. How many days would it take each working alone?

15. The area of a given rectangle is 12 sq. ft. If the length is increased by 1.5 ft. and the width by .8 ft., the area is increased by 8.9 sq. ft. Find the dimensions of the rectangle.

16. Divide $2a-a^{\frac{1}{2}}b^{-\frac{1}{2}}-3b^{-1}$ by $2a^{\frac{1}{2}}-3b^{-\frac{1}{2}}$ and verify your result by letting $a=9$ and $b=4$ in divisor, dividend, and quotient.

17. Solve $2x+y-2=0$, $4x^2+6xy+2x-6y+1=0$ and check your results.

18. Convert $\log\left(\frac{a}{b}\right)=\log a-\log b$ into a rule.

19. Find without solving whether the roots of $2.1x^2-3x+1=0$ are rational or irrational, real or imaginary, equal or unequal, positive or negative.

20. The larger of two numbers when divided by the smaller gives a quotient 3 and a remainder 8. The square of the smaller exceeds twice the larger by 11. Find the numbers.

21. Prove that the product of the first and last of any three consecutive numbers is 1 less than the square of the middle number.

22. Factor:

(1) $x^{-2}-y^{-\frac{1}{2}}$.
(2) $x^{\frac{4}{3}}-5x^{\frac{2}{3}}+6$.
(3) $a^{-3}+b^3$.
(4) $x^{2a}-y^{-6}$.
(5) $4x-y^2$.
(6) $\frac{x^2}{y^2}+2-\frac{3y^2}{x^2}$.

23. Eliminate q between $x=ap+cq$ and $y=bp-dq$.

24. Simplify and multiply $\frac{1}{2}\sqrt{8}+\sqrt{32}-\sqrt{48}$ by $3\sqrt{8}-\frac{1}{4}\sqrt{32}+2\sqrt{12}$.

25. Solve $T=\pi r(r+l)$ for r.

26. A farmer has a field 60 rd. long and 40 rd. wide. How wide a strip must he cut around the outside of the field in order to leave 5 acres uncut.

27. By use of logarithms, determine the number of years in which $100 must be at compound interest at 5 % in order to amount to $500.

28. Find the sum of the first 10 terms and also of the next 10 terms in the A. P. 30, $26\frac{2}{3}$, $23\frac{1}{3}$, ...

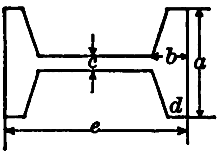
29. Solve $(a-b)x^2-a^2x+a^2b=0$ for x and check your answers.

30. Compute $\frac{3\sqrt{3}-4}{4\sqrt{3}-5}$ in the shortest way and to the nearest hundredth.

31. Solve graphically $2x^2-11x+5=0$.

32. Draw graphs of $x^2+y^2=9$, and $6x-4y=24$. Tell the algebraic meaning of the fact that the graphs do not intersect.

33. One jar contains an acid 92 % pure and another jar contains the same kind of acid 65 % pure. How much must be taken from each jar to make two gallons 80 % pure?



34. Obtain a formula for the area of the adjoining figure.

35. If each stroke of an air pump removes $\frac{2}{3}$ of the air in a receiver, what fraction of the air will be left in the receiver after 10 strokes?

36. By use of the formula $K=\sqrt{s(s-a)(s-b)(s-c)}$, find the area of the triangle whose sides are 67.2 yd., 87.8 yd., 51.4 yd. Compute by the use of logarithms.

37. Solve $x^2+xy+2y^2=74$ and $2x^2+2xy+y^2=73$.

38. A reservoir has a supply pipe A and an exhaust pipe B. The pipe A takes 4 min. less time to fill the reservoir than B takes to empty it. When both are open the reservoir is filled in 24 min. Find the time needed for A to fill the reservoir when B is closed.

39. Solve $\frac{1}{x}+\frac{1}{y}=5$ and $x-y=.3$ and check.

40. A steamer makes a trip of 3240 miles. The time of the trip would have been a day and a half less if her speed had been 2 miles per hour more. Find the time of the trip.

41. Using suitable letters, obtain a formula for the volume of a solid composed of a cylinder with a hemisphere attached at each end.

42. In the year 1913, before the great war began, the per cents of the world's coal output produced by the leading coal-producing countries were as follows: United States, 38; British Empire, 26; Germany, 25; all other countries together, 11. Graph these facts in an appropriate way.

43. Simplify (1) $(.09)^{-\frac{3}{2}}+(.064)^{\frac{2}{3}}$.

(2) $(x^m)^{n-1}.\ (x^n)^{m+1}.\ (x^m)^{1-2n}$.

44. Rationalize the denominator of $\frac{b}{\sqrt[3]{2a}}$.

45. Simplify $\left(\frac{1}{1+x}+\frac{x}{1-x}\right)\div\left(\frac{1}{1-x}-\frac{x}{1+x}\right)$ and check.

46. A 20-lb. mass of gold and silver alloy when immersed in water weighed only 18.276 lb. If gold loses .051 of its weight and silver .095 when thus weighed, how many pounds of each were in the alloy?

47. What is the shortest way of determining whether correct results have been obtained in solving a quadratic equation? By use of your method determine whether .25 and 1.5 are the roots of $4x^2-7x+1.5=0$.

48. Find the value of k which will make the roots equal in $4x^2-2(k+3)x+k^2=0$.

49. Simplify $\frac{2x^2}{(1-x^2)^{\frac{3}{2}}}+\frac{1}{(1-x^2)^{\frac{1}{2}}}$.

50. What must be the value of n in order that $(2a+n)\div(3n+69a)$ may be equal to $\frac{1}{33}$, when $a=\frac{1}{3}$?

51. The hypotenuse of a right triangle is 20. The sum of the other two sides is 28. Find the length of the sides.

52. Find to 3 decimal places the value of $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}-1}$.

53. Simplify $\sqrt{(m-n)^2a}+\sqrt{(m+n)^2a}-\sqrt{am^2}+\sqrt{a(1-m)^2}$.

54. The United States 5-cent piece (or nickel) is 75 % copper and 25 % nickel. If a mass of nickel and copper weighing 80 pounds is 90 % copper, how many pounds of nickel must be added to it to make it ready for coinage into 5-cent pieces?

55. Solve $\sqrt{x}+\sqrt{3-\sqrt{3}\,x+x^2}=\sqrt{3}$.

56. Find the square root of the product of x^2-1, x^2-3x+2, and x^2-x-2 in the shortest way.

57. Evaluate $\frac{4\sqrt{12.5}-2\sqrt{.5}}{6\sqrt{18}}$, without extracting any root.

58. From $\frac{3}{5-2\sqrt{3}}$ subtract $\frac{1}{3+\sqrt{3}}$ and express the result as a fraction having a rational denominator.

59. Multiply $\sqrt[4]{m^{-3}}-\sqrt{m^{-1}}\sqrt[4]{n}+\sqrt[4]{m^{-1}}\sqrt{n}-\sqrt[4]{n^3}$ by $m^{-\frac{1}{4}}+n^{\frac{1}{4}}$.

60. Find the algebraic expression which, when divided by $x^2-2\,x+1$, gives a quotient of $x^2+2\,x+1$ and a remainder of $x-1$.

61. Find the numerical value of $(5.1)^{\frac{2}{3}}$ to 4 decimal places.

62. Separate 200 into three such parts that the first divided by the second gives 2 for quotient and 2 for remainder; and the second divided by the third gives 4 for quotient and 1 for remainder.

63. Simplify $(3\sqrt[4]{\tfrac{1}{9}})^2$.

64. Eliminate l between the equations: $s=\frac{wl^2}{8\,t}$ and $l'=l+\frac{8\,s^2}{3\,l}$.

65. Simplify $\frac{a^{-2}-b^{-2}}{a^{-1}-b^{-1}}$. Also $\frac{(1-a)^{-1}+(1+a)^{-1}}{(1-a)^{-1}-(1+a)^{-1}}$.

66. Rationalize the denominator of $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}$ and find the value of the result when $a=\frac{4}{5}$, $b=20$, and $c=5$.

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